Rethink Deep Learning with Invariance in Data Representation

A Tutorial at The Web Conference 2025 in Sydney (WWW 2025)

Shuren Qi[1], Fei Wang[2], Tieyong Zeng[1], and Fenglei Fan[3]

[1] CMAI, IMS, Department of Mathematics, The Chinese University of Hong Kong, HK
[2] Weill Cornell Medicine, Department of Population Health Sciences, Cornell University, USA
[3] Department of Data Science, City University of Hong Kong, HK

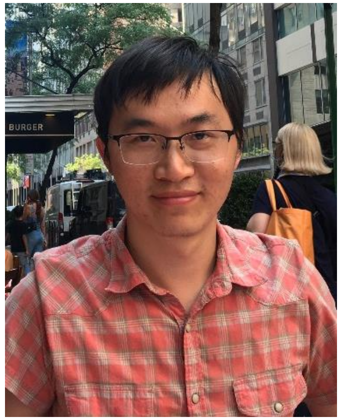
13:30 - 16:30, Tuesday, April 29, 2025
Room C3.4, ICC Sydney, Australia

Organizer and Presenter

Shuren Qi
Postdoc, CUHK
shurenqi@cuhk.edu.hk
Presenter

Fei Wang
Professor, Cornell
few2001@med.cornell.edu
Organizer

Tieyong Zeng
Professor, CUHK
zeng@math.cuhk.edu.hk
Organizer



Fenglei Fan
AP, CityU
fenglfan@cityu.edu.hk
Main Organizer

Tutorial Homepage

Tutorial proposal, slides, reading list, video, and more materials available at

https://shurenqi.github.io/wwwtutorial/

Rethink Deep Learning with Invariance in Data Representation

Shuren Qi
shurenqi@cuhk.edu.hk
The Chinese University of Hong Kong
Hong Kong, China

Fei Wang
few2001@med.cornell.edu
Cornell University
New York, United States

Tieyong Zeng
zeng@math.cuhk.edu.hk
The Chinese University of Hong Kong
Hong Kong, China

Fenglei Fan
flfan@math.cuhk.edu.hk
The Chinese University of Hong Kong
Hong Kong, China

Abstract

Integrating **invariance** into data **representations** is a principled design in intelligent systems and web applications. Representations play a fundamental role, where systems and applications are both built on meaningful representations of digital inputs (rather than the raw data). In fact, the proper design/learning of such representations relies on priors w.r.t. the task of interest. Here, the concept of *symmetry* from the *Erlangen Program* may be the most fruitful prior — informally, a symmetry of a system is a transformation that leaves a certain property of the system invariant. Symmetry priors are ubiquitous, e.g., translation as a symmetry of the object classification, where object category is invariant under translation.

The quest for invariance is as old as pattern recognition and data mining itself. Invariant design has been the cornerstone of various representations in *the era before deep learning*, such as the SIFT. As we enter *the early era of deep learning*, the invariance principle is largely ignored and replaced by a data-driven paradigm, such as the CNN. However, this neglect did not last long before they encountered bottlenecks regarding robustness, interpretability, efficiency, and so on. The invariance principle has returned in *the era of rethinking deep learning*, forming a new field known as Geometric Deep Learning (GDL).

In this tutorial, we will give a **historical perspective** of the invariance in data representations. More importantly, we will identify those research dilemmas, promising works, future directions, and web applications.

CCS Concepts

• **Theory of computation → Theory and algorithms for application domains**; • **Computing methodologies → Artificial intelligence**.

Keywords

Pattern Recognition, Data Mining, Invariance, Symmetry, Representation, Tutorial

ACM ISBN 979-8-4007-1331-6/25/04
https://doi.org/10.1145/3701716.3715861

ACM Reference Format:
Shuren Qi, Fei Wang, Tieyong Zeng, and Fenglei Fan. 2025. Rethink Deep Learning with Invariance in Data Representation. In *Companion Proceedings of the ACM Web Conference 2025 (WWW Companion '25), April 28-May 2, 2025, Sydney, NSW, Australia.* ACM, New York, NY, USA, 4 pages. https://doi.org/10.1145/3701716.3715861

1 Topic and Relevance

The topic of this tutorial is a historical review of the invariance in data representations. The scope of this tutorial covers 1) the invariance in the era before deep learning, on old-fashioned invariant designs from various hand-crafted representations; 2) the invariance in the early era of deep learning, on the slump of the invariance principle and the success of the data-driven paradigm; 3) the invariance in the era of rethinking deep learning, on the revival of the invariance principle and the emergence of geometric deep learning as a way to bridge the research gap. For the depth within each era, the research dilemmas, promising works, future directions, and web applications will be sorted out. *More details are expanded in **Section 2**.*

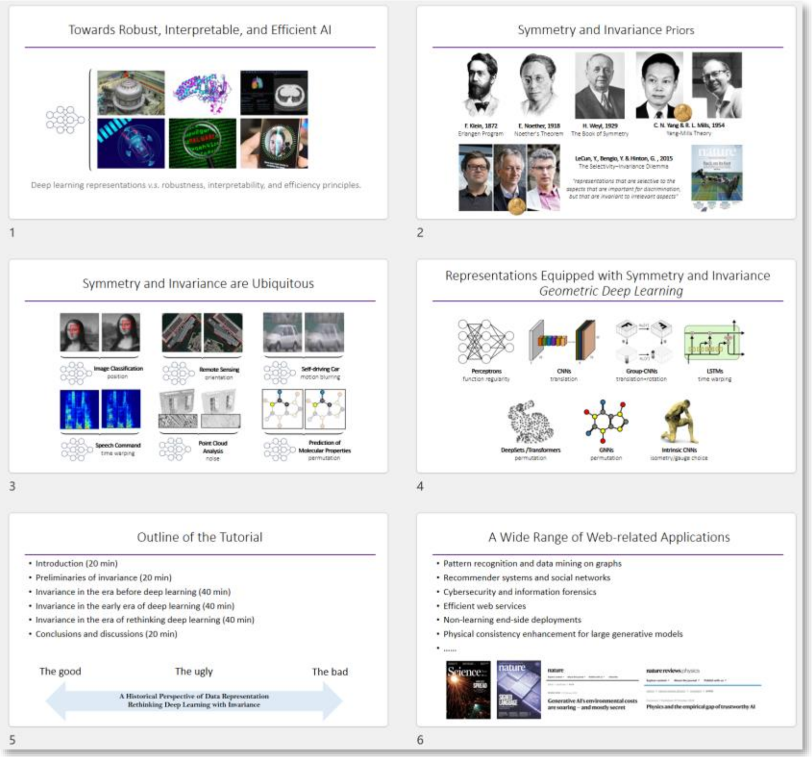
The presenters are qualified for a high-quality introduction to the topic. We have extensive research experience and strong publication records in representation backbones and downstream applications of pattern recognition and data mining. *More details are expanded in **Section 3**.*

This tutorial is timely, due to the general limitations of today's intelligent systems and their web applications with respect to being only data-driven. Also, the invariance perspective (technology focus) and the historical perspective (broad horizons) are rarely seen in the tutorial tracks of related conferences.

This tutorial is relevant to the Web Conference. From a technological perspective, representations play a fundamental role in intelligent systems and their wide range of downstream web applications. From a practical perspective, the currently popular data-driven paradigm has led to bottlenecks in intelligent systems and their web applications, regarding robustness, interpretability, efficiency, and so on. Understanding invariance in data representations is helpful in facilitating better web applications.

2 Content

Over the past decade, deep learning representations, e.g., convolutional neural networks (CNN) and transformer, have led to breakthrough results in numerous artificial intelligence (AI) tasks, e.g., processing human perceptual information, playing board games,



Tutorial Outline

- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

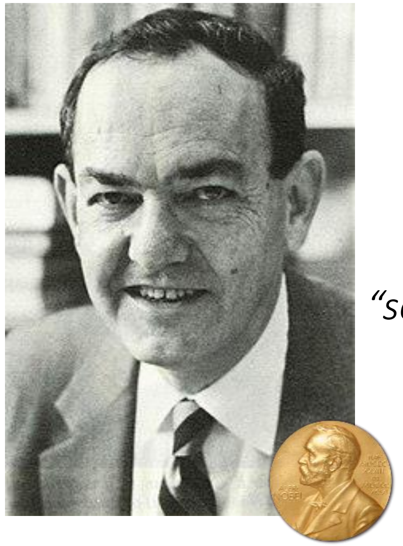
Tutorial Outline

- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

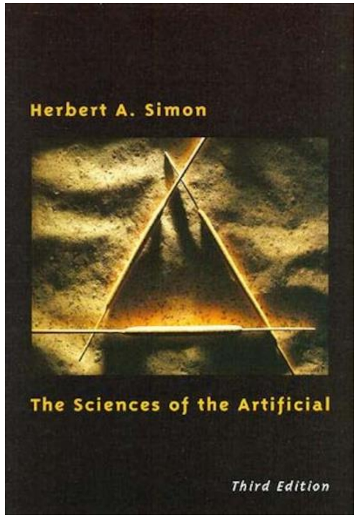
Deep (Representation) Learning,
A Big Bang Moment For AI

Data Representation



H. Simon, 1969
The Sciences of the Artificial

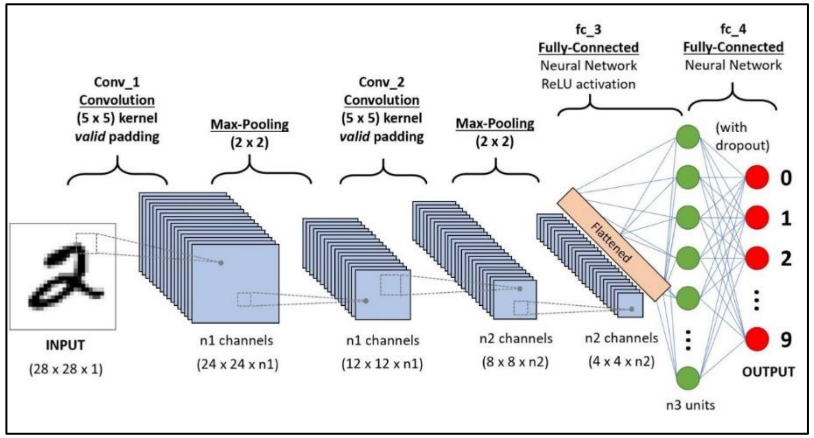
"solving a problem simply means representing it so as to make the solution transparent"

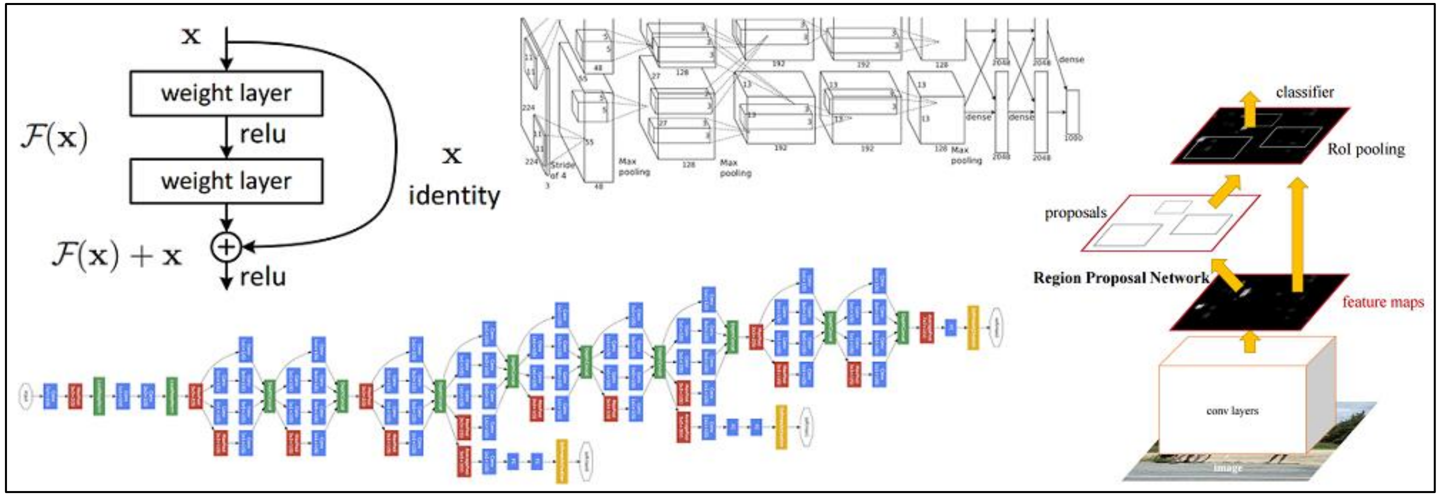


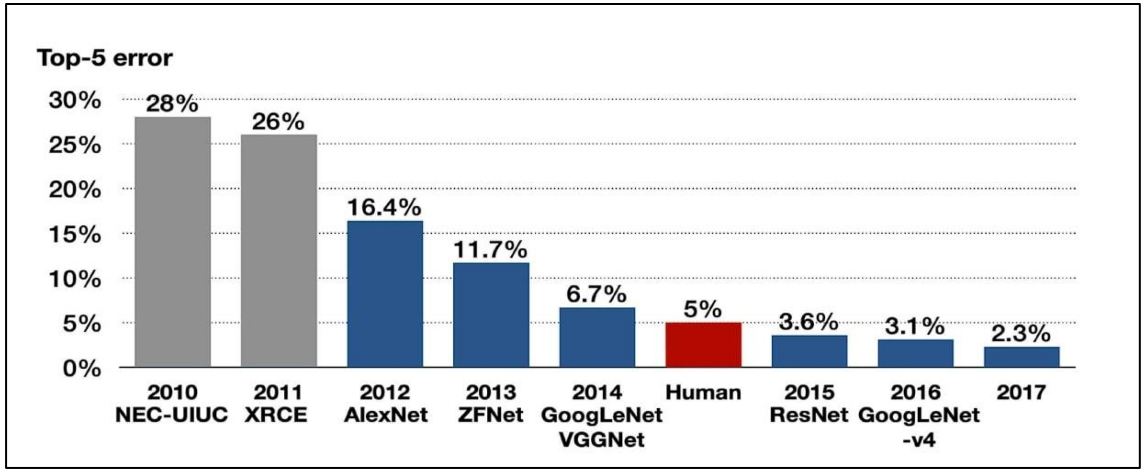
Data ⇨ **Representation** ⇨ Knowledge discovery ⇨ Application

- H Simon. *The Sciences of the Artificial (Third edition)*. MIT Press, 1996.

Processing Human Perceptual Information



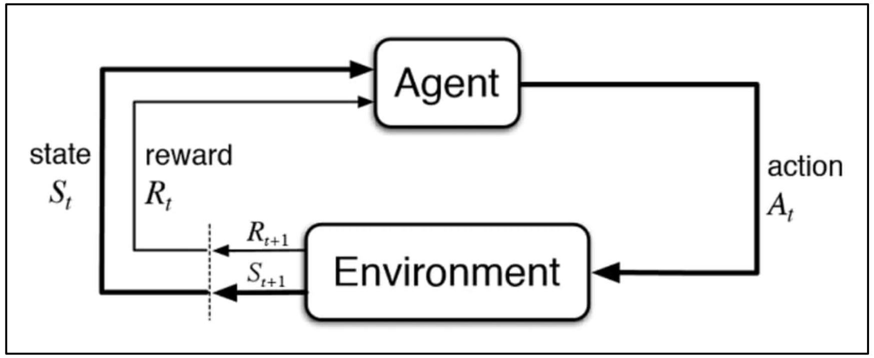


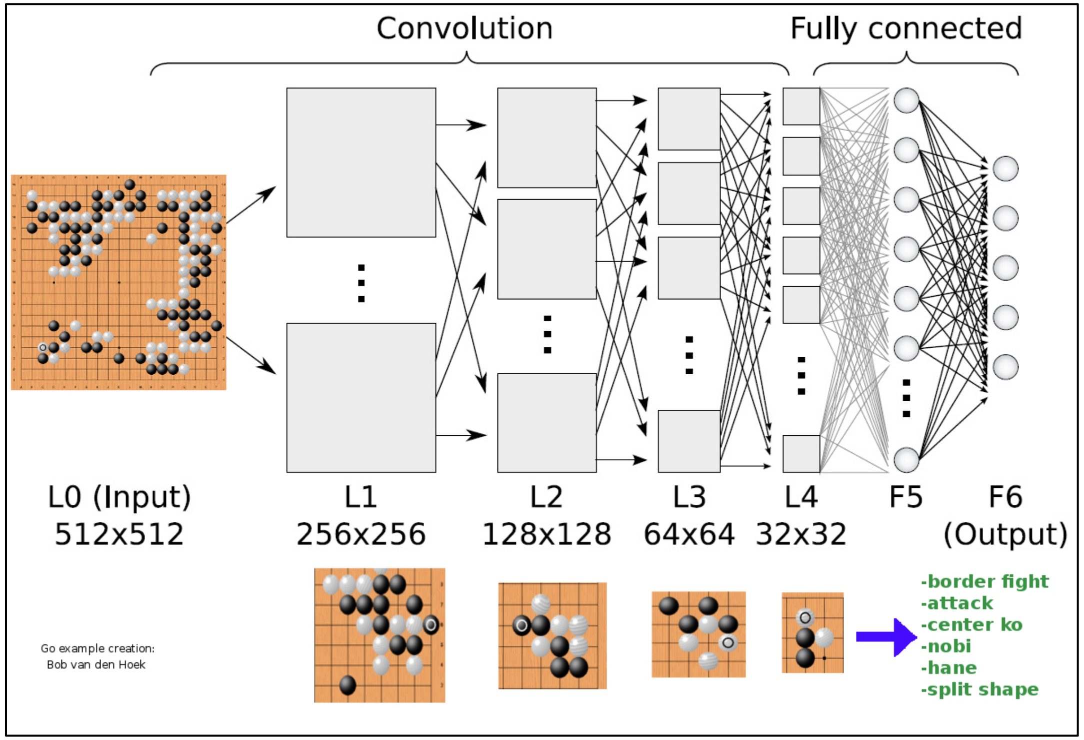


- J Deng, W Dong, R Socher, et al. ImageNet: A large-scale hierarchical image database. *CVPR,* 2009.

Playing Board Games

AlphaGo





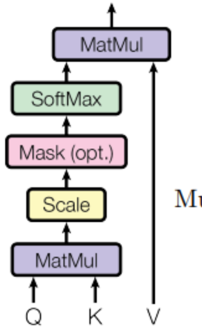
- D Silver, J Schrittwieser, K Simonyan, et al. Mastering the game of go without human knowledge. *Nature,* 2017.

Generating Realistic Media

ChatGPT

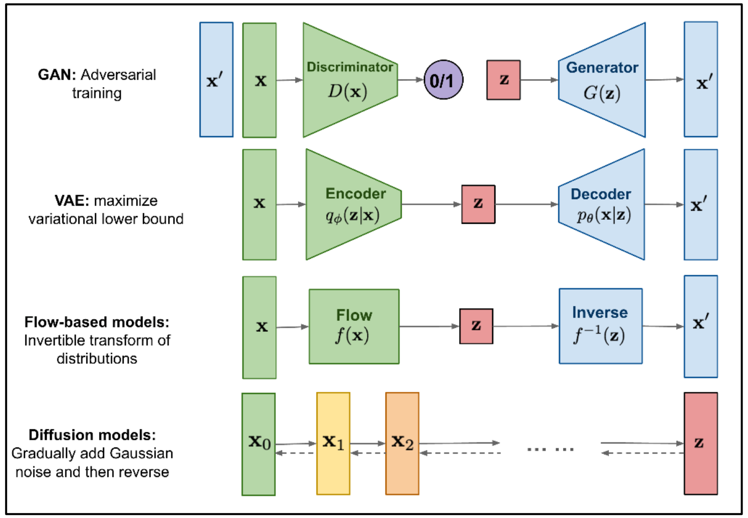
DALL·E 3

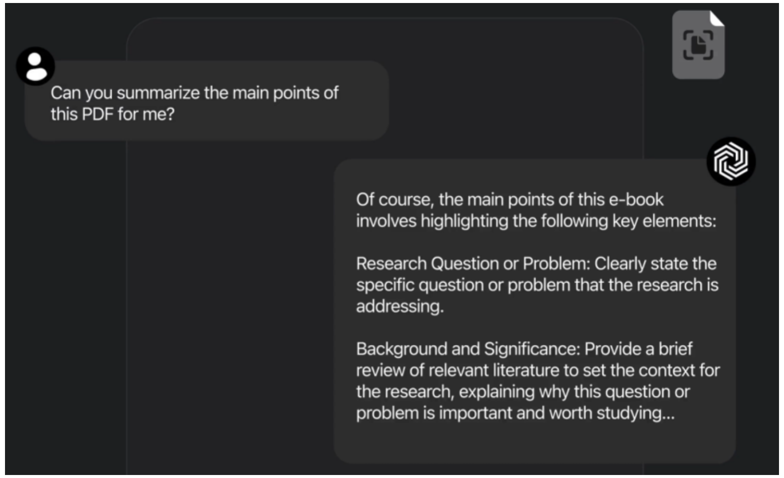
OpenAI SORA



$$\mathrm{Attention}(Q,K,V) = \mathrm{softmax}\left(\frac{QK^T}{\sqrt{d_k}}\right)V$$

$$\mathrm{MultiHead}(Q,K,V) = \mathrm{Concat}(\mathrm{head}_1, \ldots, \mathrm{head_h})W^O$$
$$\text{where } \mathrm{head_i} = \mathrm{Attention}(QW_i^Q, KW_i^K, VW_i^V)$$

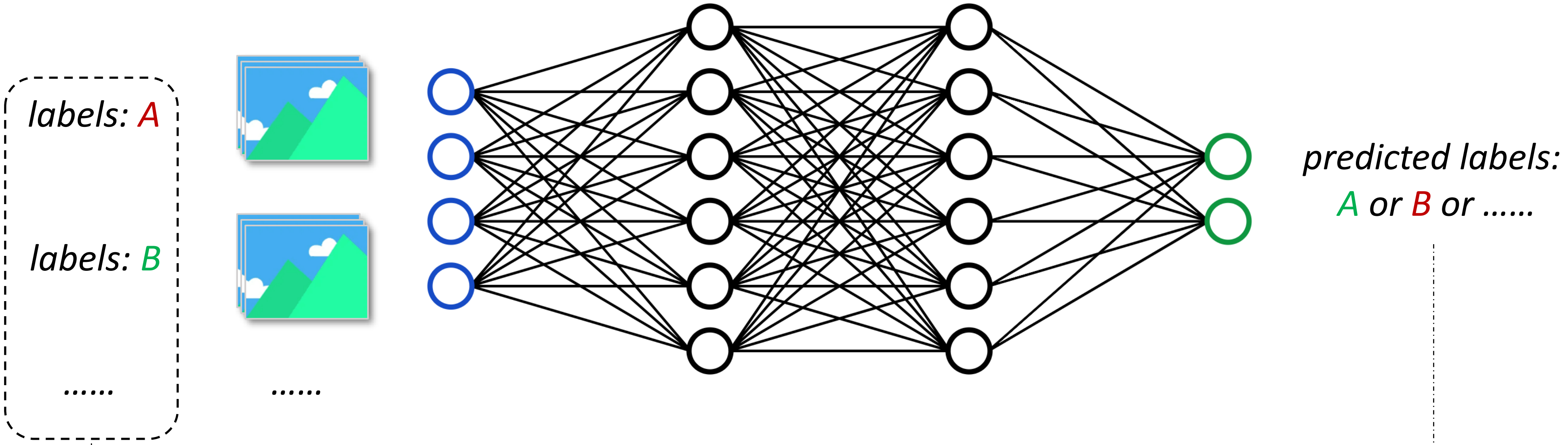




- Z Epstein, A Hertzmann, L. Herman, et al. Art and the science of generative AI. *Science*, 2023.

Empirical Risk Minimization (ERM), Behind All These Successes

Empirical Risk Minimization



$$\mathbb{E}_{\mathbf{x},\mathbf{y}\in S}[\ell(\mathbf{y}, h(\mathbf{x}))]$$

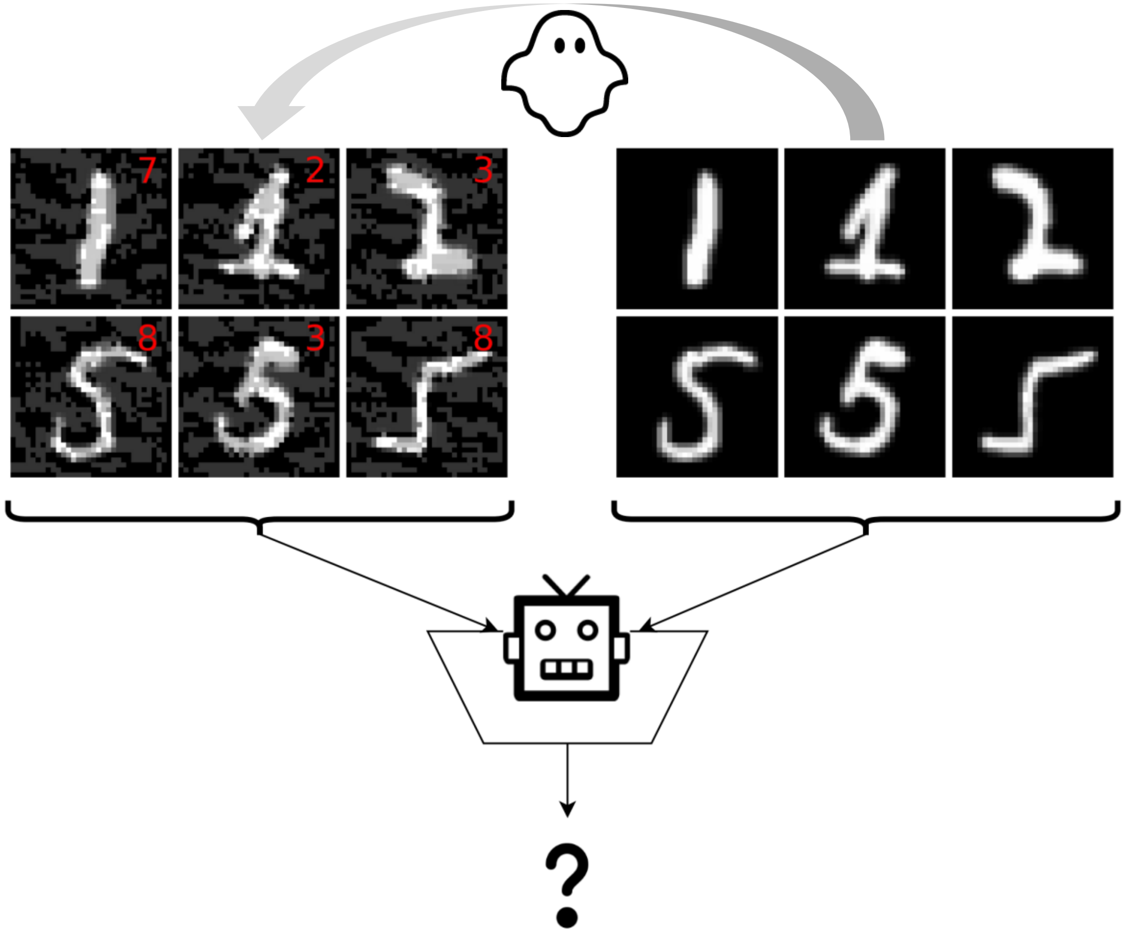
Empirical learning draws the lines between categories.

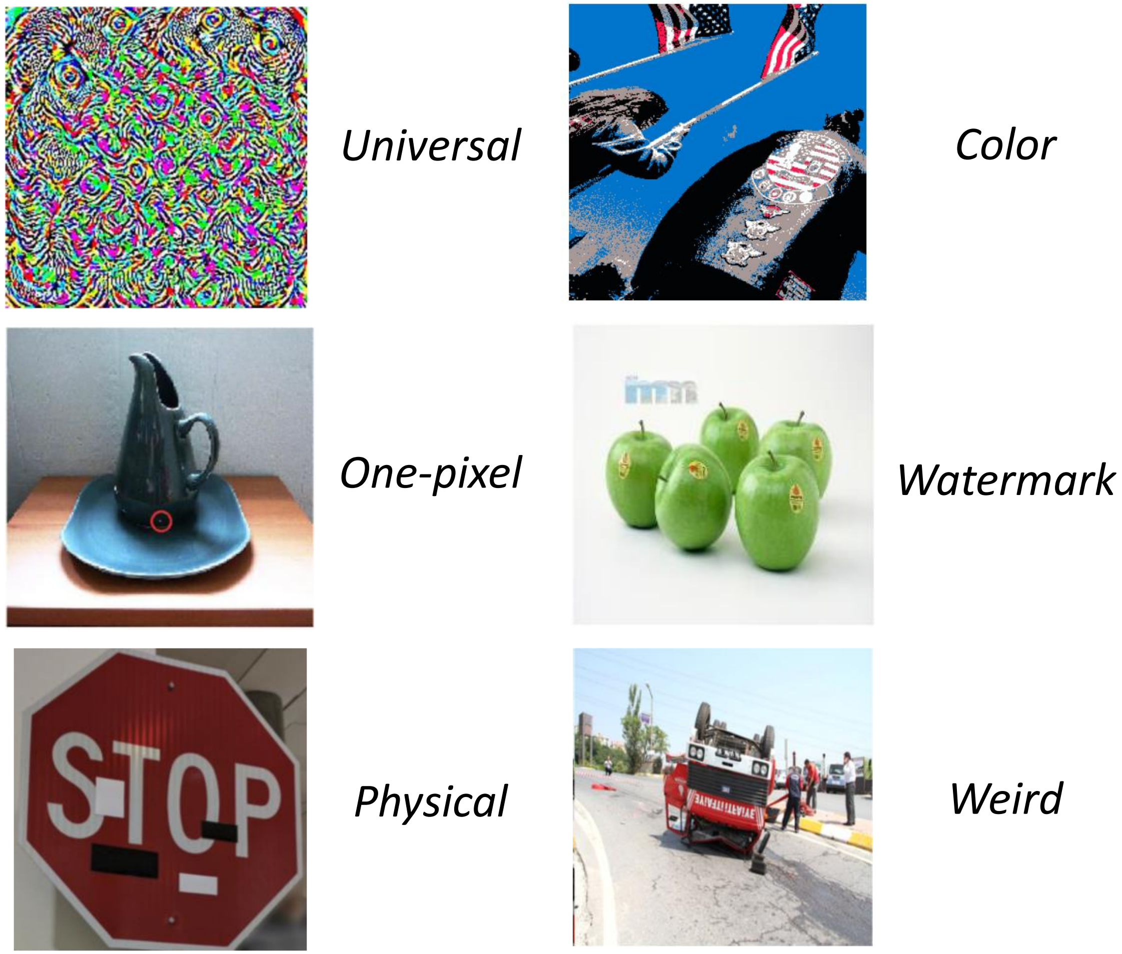
But what about robustness, interpretability, and efficiency?

- V Vapnik. Principles of risk minimization for learning theory. *NIPS,* 1991.

Robustness of Empirical Learning

- Robustness: the performance of a system is stable for intra-class variations on the input.

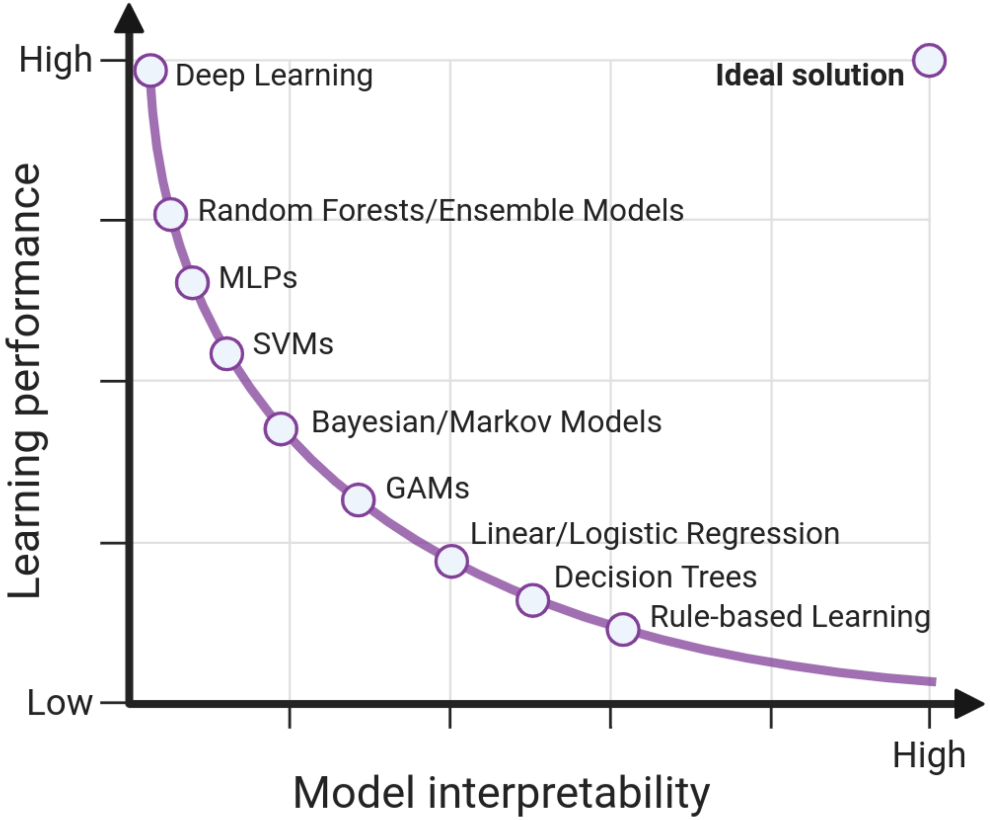


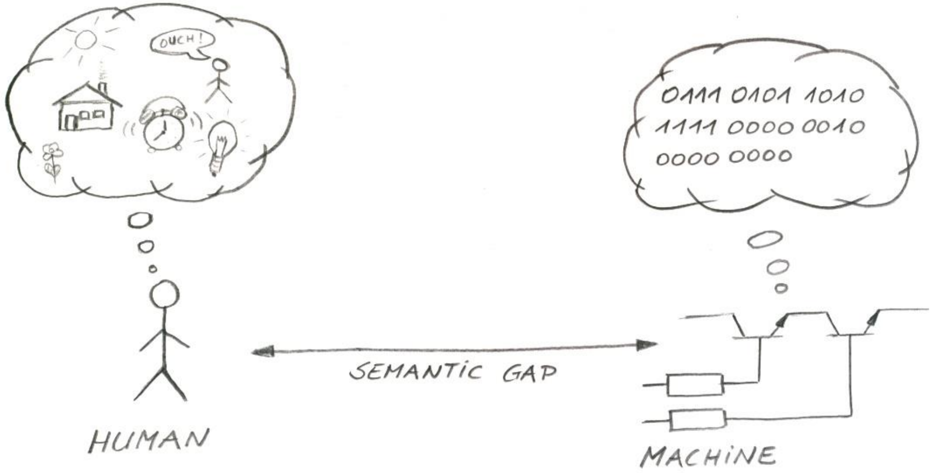


- C Buckner. Understanding adversarial examples requires a theory of artefacts for deep learning. *Nature Machine Intelligence*, 2020.

Interpretability of Empirical Learning

- Interpretability: the behavior of a system can be understood or predicted by humans.

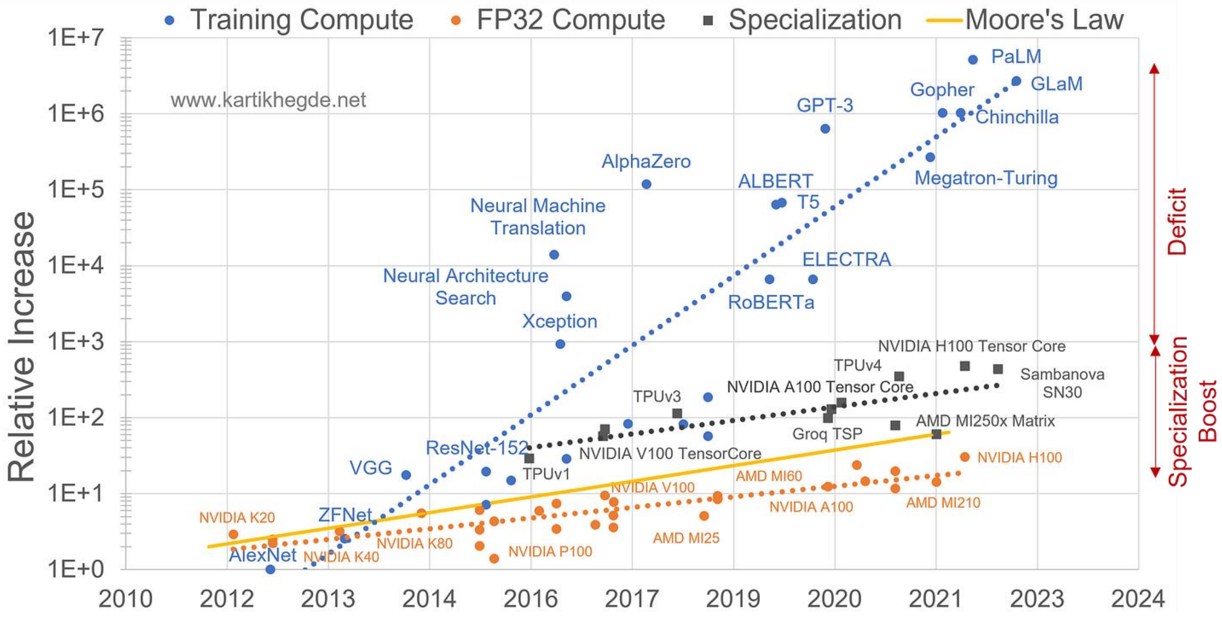


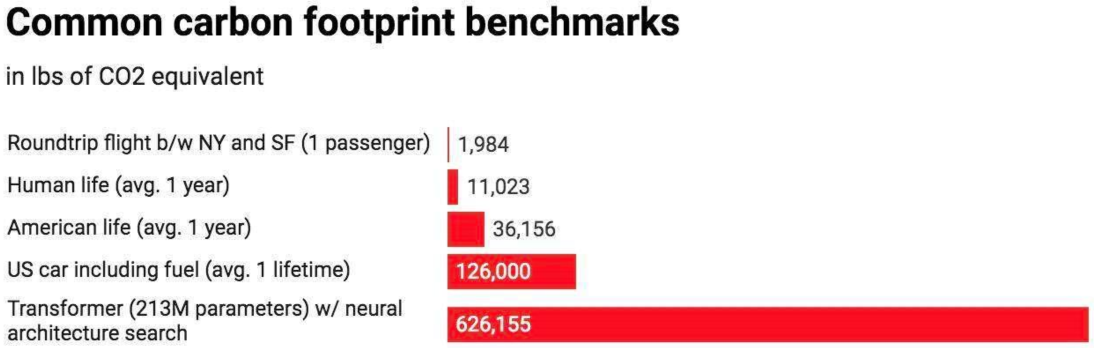


- X Li, C Cao, Y Shi, et al. A survey of data-driven and knowledge-aware explainable AI. *TKDE,* 2020.

Efficiency of Empirical Learning

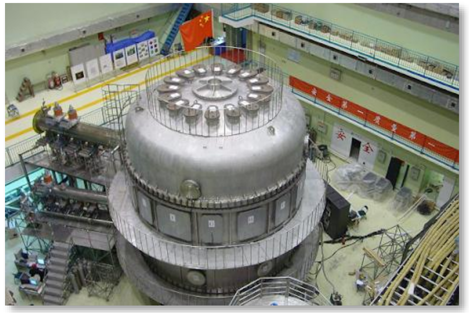
- Efficiency: the real-time availability and energy cost during human-computer interaction.



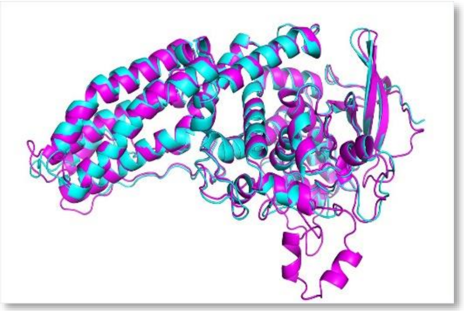


- E Strubell, A Ganesh, A McCallum, et al. Energy and policy considerations for modern deep learning research. *AAAI*, 2020.

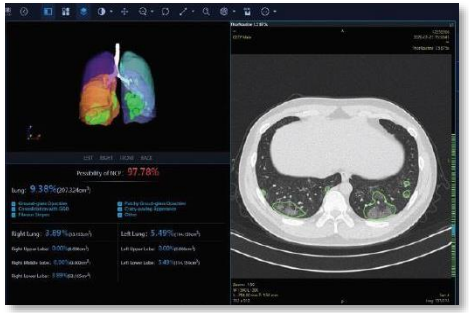
When Moving Towards Trustworthy AI



Tokamak Control



Drug Discovery



Imaging Diagnostics

Automatic Driving

Cyber Security

Biometrics

Empirical learning *v.s.* robustness, interpretability, efficiency…

- H Liu, M Chaudhary, H Wang. Towards trustworthy and aligned machine learning: A data-centric survey with causality perspectives. *arXiv preprint arXiv:2307.16851*, 2023.

A Foundational Prior Underlying Both Natural World And AI Systems

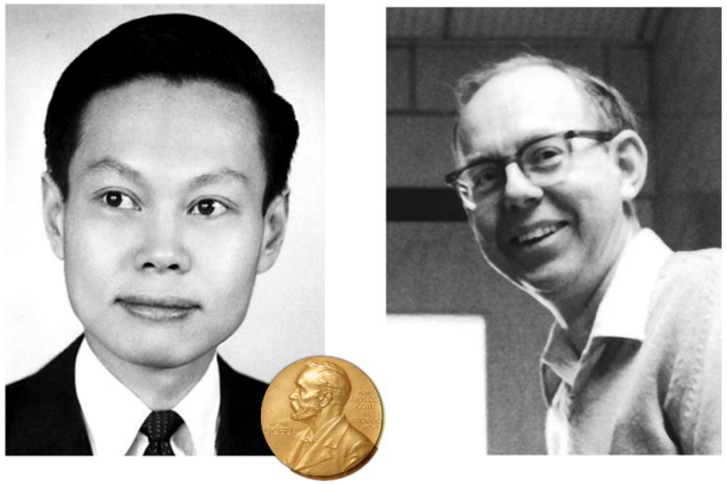
Invariance/Symmetry in Natural World

- A symmetry of a system is a transformation that leaves a certain property invariant.

F. Klein, 1872
Erlangen Program

E. Noether, 1918
Noether's Theorem

H. Weyl, 1929
The Book of Symmetry



C. N. Yang & R. L. Mills, 1954
Yang-Mills Theory

- F Klein. A comparative review of recent researches in geometry. *Bulletin of the American Mathematical Society*, 1893.
- H Weyl. *Symmetry.* Princeton University Press, 2015.

Invariance/Symmetry in AI Systems

- An AI system is a digital modeling of the physical systems in the natural world.

Y. LeCun, Y. Bengio & G. Hinton, 2015, Deep learning, Nature

The Selectivity–Invariance Dilemma:
“representations that are selective to the aspects that are important for discrimination, but that are invariant to irrelevant aspects”

- Y LeCun, Y Bengio, G Hinton. Deep learning. *Nature*, 2015.
- Y Bengio, A Courville, P Vincent. Representation learning: A review and new perspectives. *TPAMI*, 2013.

How Invariance/Symmetry Helps Robustness, Interpretability, Efficiency

- Perfect robustness — the performance of the AI system remains invariant with respect to the transformations of interest.
- Interpretable concept — humans and AI systems share a basic concept that allows humans to predict AI behavior on transformations of interest.
- Structural efficiency — AI systems no longer need to memorize non-discriminative data variants.

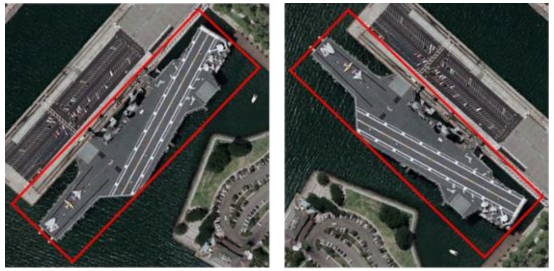
Tutorial Outline

- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

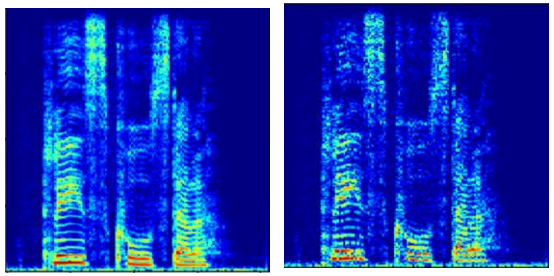
Invariance/Symmetry is Ubiquitous in AI Tasks

Image Classification
position

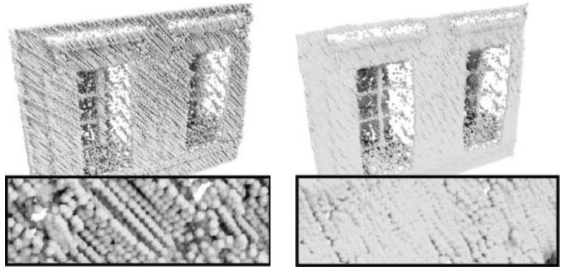


Remote Sensing
orientation

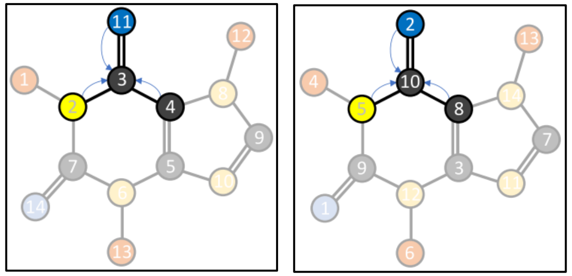
Self-driving Car
motion blurring



Speech Command
time warping



Point Cloud Analysis
noise

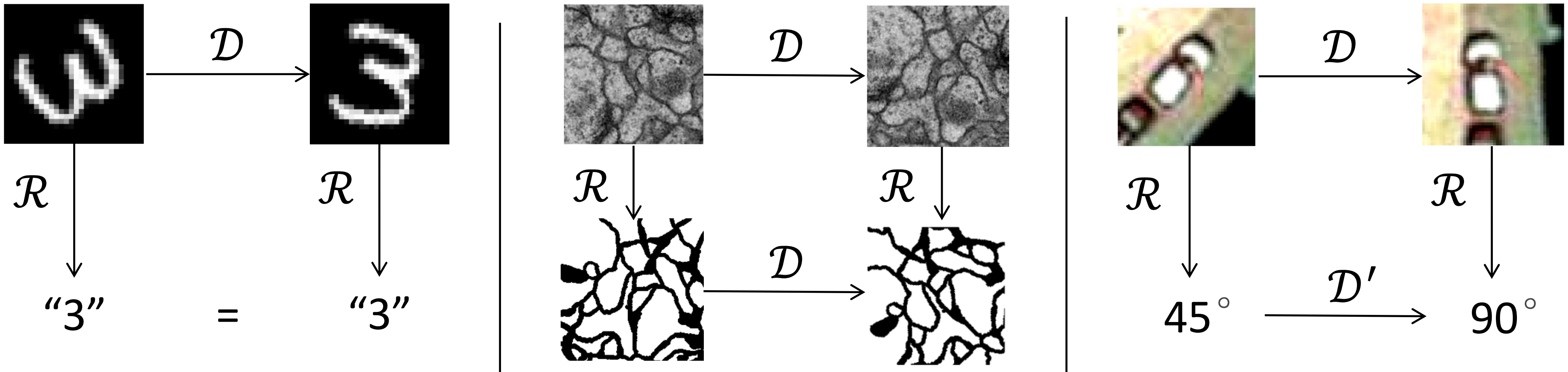


Prediction of Molecular Properties
permutation

A Formalization of Invariance/Symmetry (in Representation)

- Invariance: $\mathcal{R}\big(\mathcal{D}(f)\big) \equiv \mathcal{R}(f)$
- Equivariance: $\mathcal{R}\big(\mathcal{D}(f)\big) \equiv \mathcal{D}\big(\mathcal{R}(f)\big)$
- Covariance: $\mathcal{R}\big(\mathcal{D}(f)\big) \equiv \mathcal{D}'\big(\mathcal{R}(f)\big)$
- $\mathcal{R}$ is a representation, $\mathcal{D}$ is a degradation, and invariance/equivariance is a special case of covariance with $\mathcal{D}' = \mathrm{id}/\mathcal{D}$

- K Lenc, A Vedaldi. Understanding image representations by measuring their equivariance and equivalence. *CVPR*, 2015.



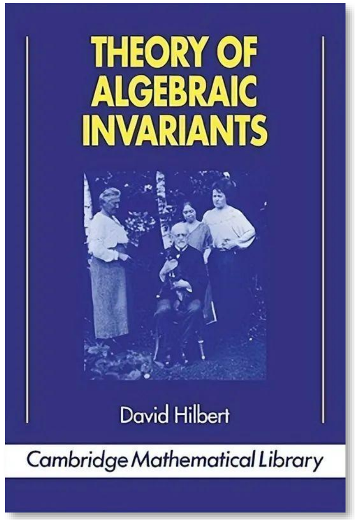
Invariance
$\mathcal{D}$ = rotate, $\mathcal{R}$ = classifier

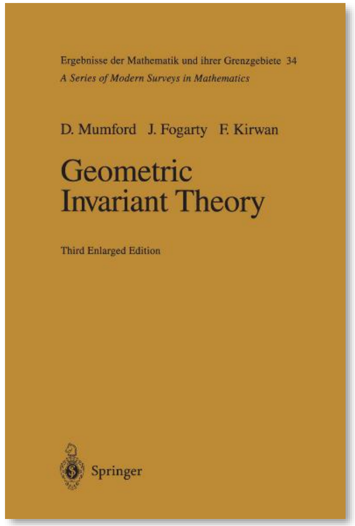
Equivariance
$\mathcal{D}$ = rotate, $\mathcal{R}$ = detector

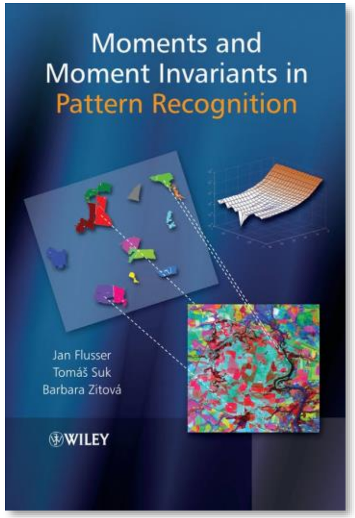
Covariance
$\mathcal{D}$ = rotate, $\mathcal{R}$ = estimator

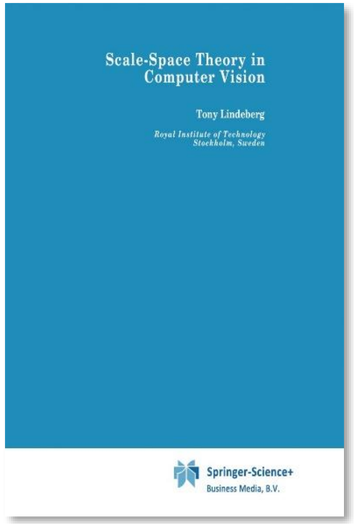
A History of Invariance/Symmetry (in Representation)

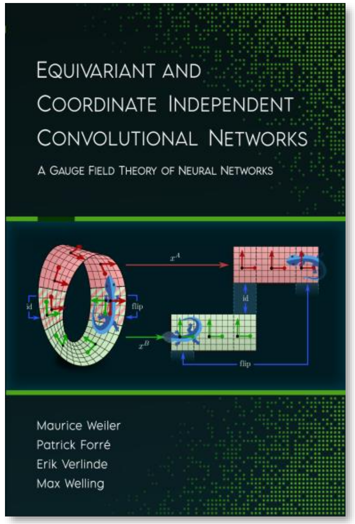
Algebraic Invariants	Geometric Invariants	Moment Invariants	Multiscale and Wavelet	CNN to Geometry Deep Learning
1840s	*1960s*	*2000s*	*2010s*	*2020s*
Hilbert Cayley Klein…	Mumford	Flusser	Lowe Lindeberg Mallat…	LeCun Bronstein…



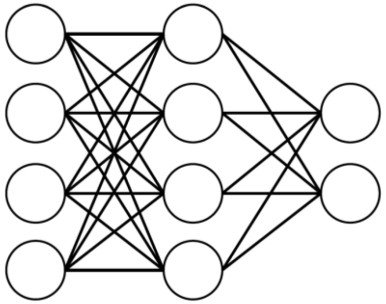




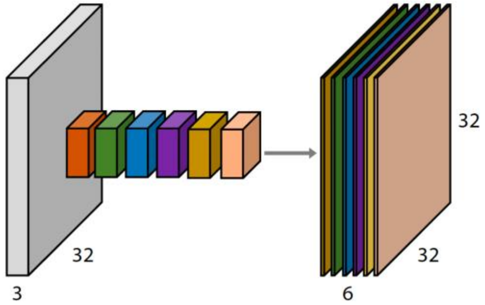




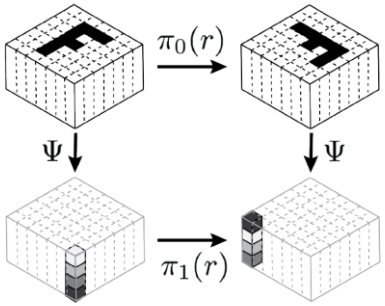
Rethinking Representations by Invariance/Symmetry
Geometric Deep Learning



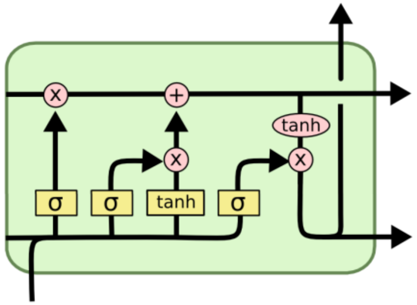
Perceptrons
function regularity



CNNs
translation

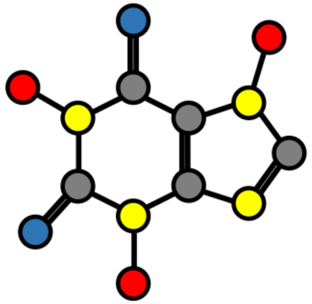


Group-CNNs
translation+rotation



LSTMs
time warping

DeepSets /Transformers
permutation



GNNs
permutation

Intrinsic CNNs
isometry/gauge choice

- MM Bronstein, J Bruna, Y LeCun, et al. Geometric deep learning: going beyond euclidean data. *IEEE Signal Processing Magazine*, 2017.

What I Did With My Collaborators In The Process Of Invariance?

Our Contributions

	Moment Invariants	Multiscale and Wavelet	CNN to Geometry Deep Learning
	2000s	*2010s*	*2020s*
	Flusser	Lowe Lindeberg Mallat…	LeCun Bronstein…
Assumption	Global	Local	Hierarchical
Invari./Discri.	Invariance	Mid?	Discriminability
Robust./Interp./Effic.	Good	Mid?	Not Good
Our Contributions	CSUR'23 TPAMI'24 TIFS review	TPAMI'23 USENIX Security'25 TIP'25	ICCV'25 TIP'25

Tutorial Outline

- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly


Time for
a Break!!

Tutorial Outline

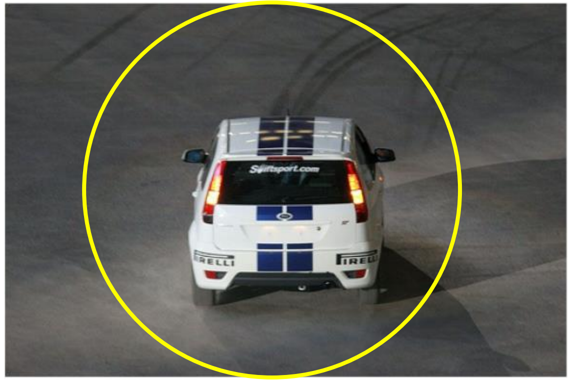
- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

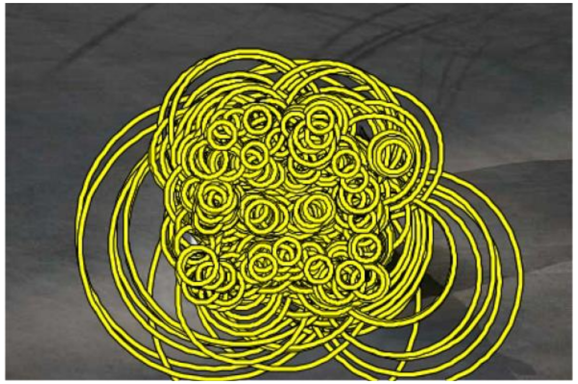
Invariance in The Era Before Deep Learning

- In the era before deep learning, data representations were almost always designed by experts manually, driven by knowledge in math, physics, signal processing, and computer vision.
- Depending on the spatial scope of the action, these representations can be classified as global, locally sparse and locally dense. Such assumptions are different and lead to different realizations of invariance.

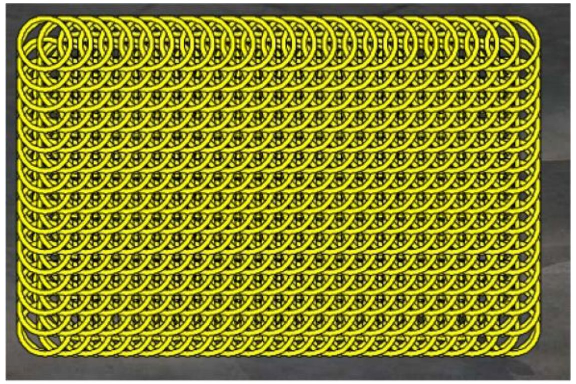
Original Image



Global Representation



Locally Sparse Representation

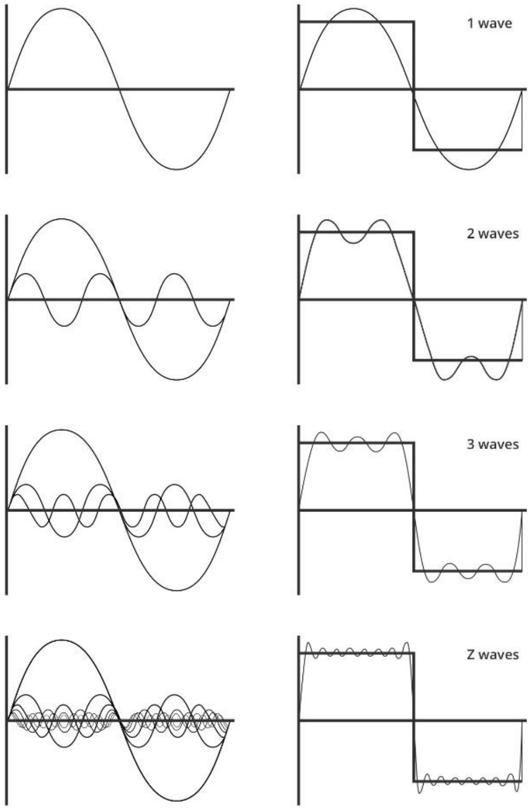


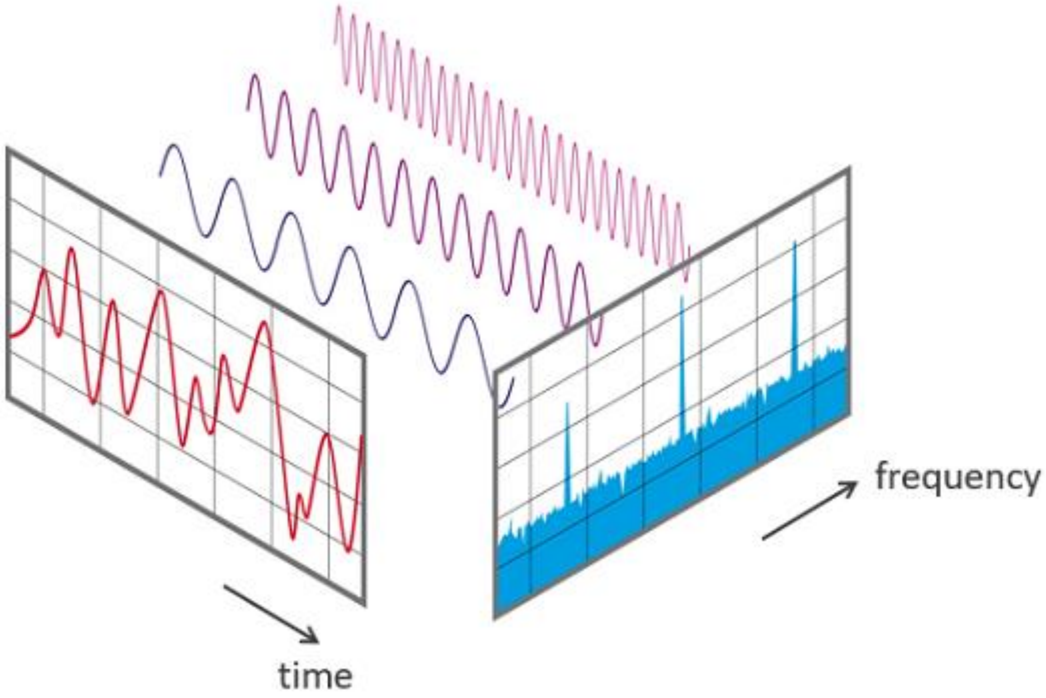
Locally Dense Representation

- K Mikolajczyk, C Schmid. A performance evaluation of local descriptors. *TPAMI*, 2005.

Global Representations: Fourier Transform

- **Fourier Transform** is a tool that rewrite a (continuous and smooth) function as a (coefficient-weighted) sum of sine/cosine functions.

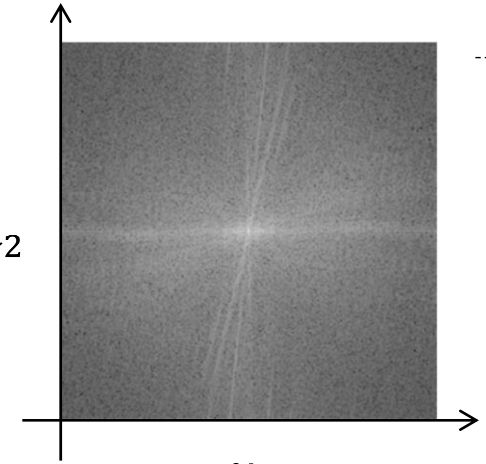




Global Representations: Fourier Transform

- Image, as a 2D function, can also be rewritten as a sum of 2D sine/cosine functions:

u_2

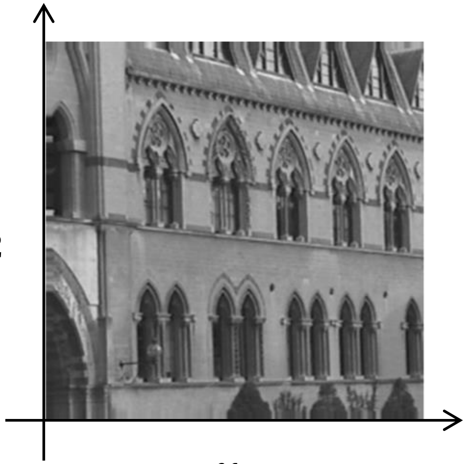


u_1

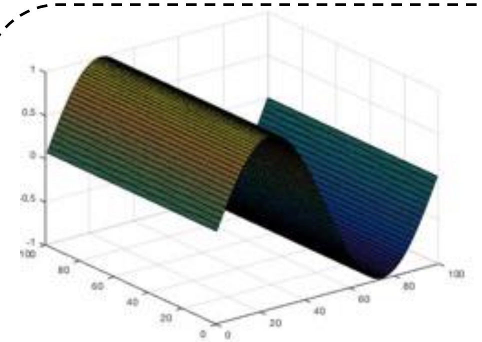
$$\mathcal{F}(f)(\mathbf{u}) \equiv F(\mathbf{u}) = \int_{\mathbb{R}^d} \boxed{e^{-2\pi i \mathbf{u}\mathbf{x}}} f(\mathbf{x}) \mathrm{d}\mathbf{x}$$

$$\mathcal{F}^{-1}(F)(\mathbf{x}) = \int_{\mathbb{R}^d} e^{2\pi i \mathbf{u}\mathbf{x}} F(\mathbf{u}) \mathrm{d}\mathbf{u} = f(\mathbf{x})$$

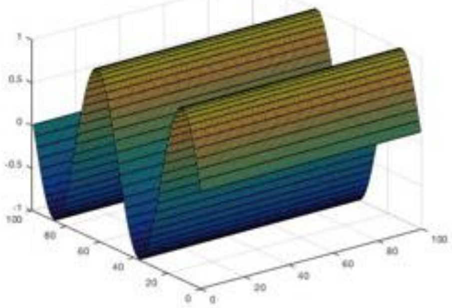
x_2



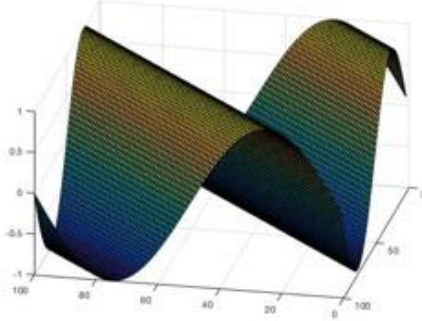
x_1



$u_1 = 1, u_2 = 0$



$u_1 = 0, u_2 = 2$



$u_1 = 1, u_2 = 1$

… …

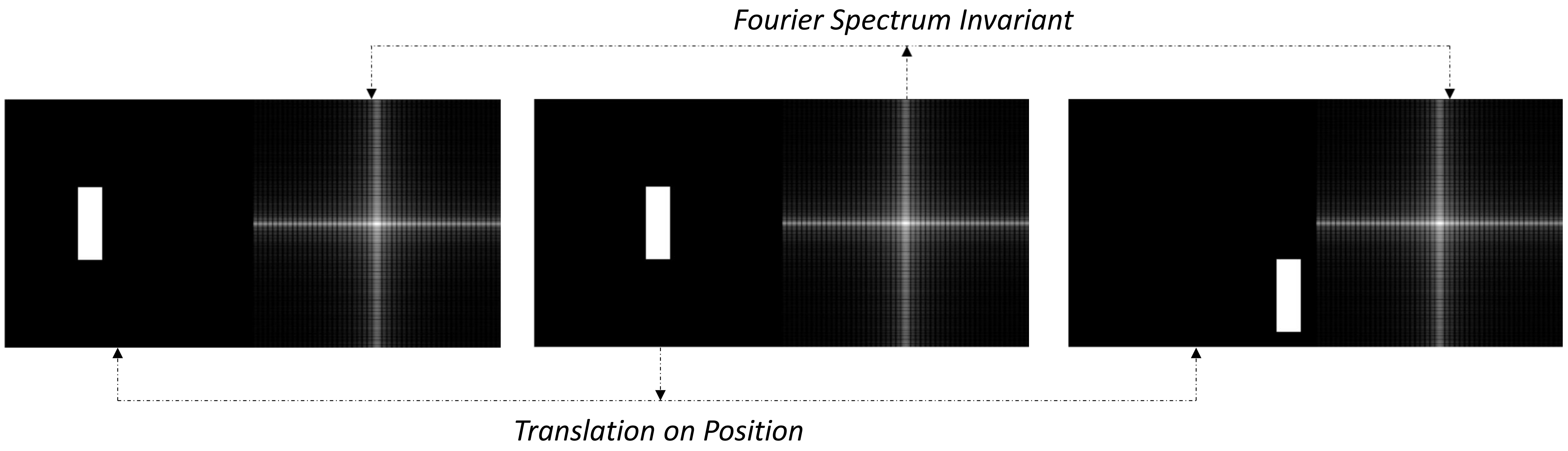
So, How About Invariance?

Translation Invariance of Fourier Transform

- Translating the function leads to multiplying the Fourier transform by a phase factor:

$$\mathcal{F}(f(\mathbf{x}-\mathbf{t}))(\mathbf{u}) = \boxed{e^{-2\pi i \mathbf{u}\mathbf{t}}}\mathcal{F}(f(\mathbf{x}))(\mathbf{u})$$

- As a consequence, the absolute values of Fourier transform are invariant to translation.



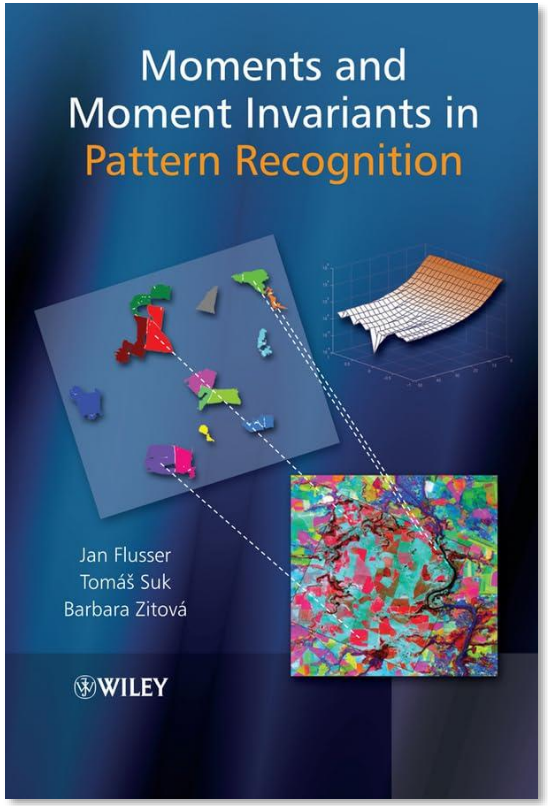
Can Global Invariance Be Generalized To Other Geometric Transformations?

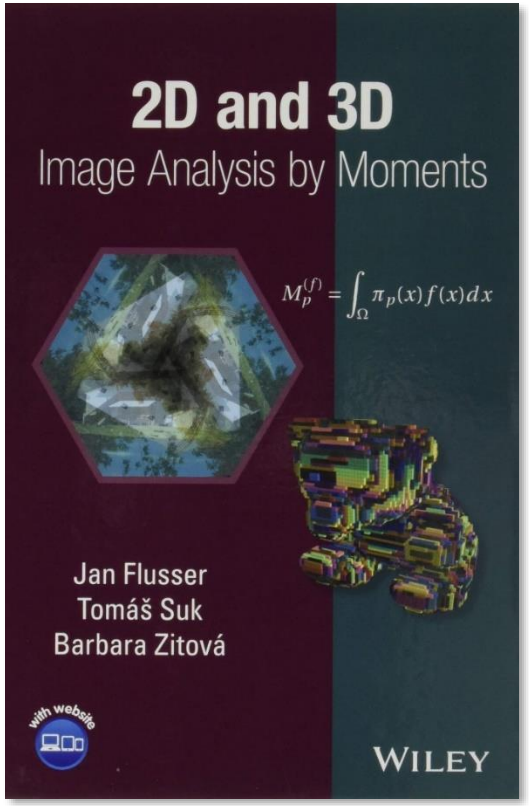
Global Representations: Moment Invariants

- **Moment Invariants** are similar to Fourier transforms in that they also rewrite the function as a (coefficient-weighted) sum of basis functions, but with a different purpose — more generalized invariants.



J. Flusser, B. Zitova, & T. Suk, 2009
Moment Invariants





- J. Flusser, B. Zitova, T. Suk. *Moments and Moment Invariants in Pattern Recognition*. John Wiley & Sons, 2009.

Moments as a Generic Form of Global Representation

- **Moments** have a very simple definition, and is in fact a generic form of the global representation:

$$M_{\mathbf{p}}^{(f)} = \int_{\Omega} \pi_{\mathbf{p}}(\mathbf{x}) f(\mathbf{x}) \mathrm{d}\mathbf{x}$$

- Here, the core is how such basis functions π are designed so that more generalized invariants I can be derived from the corresponding moments M by a certain operation ω.

$$\begin{tikzcd}
f \arrow[rrrr, phantom, bend left, "\mathcal{R}"] \arrow[r] \arrow[d, "\mathcal{D}"'] & \pi \arrow[r] & M \arrow[r] & \omega \arrow[r] & I \arrow[d, "="] \\
f' \arrow[r] & \pi \arrow[r] & M' \arrow[r] & \omega \arrow[r] & I
\end{tikzcd}$$

- J. Flusser, B. Zitova, T. Suk. *Moments and Moment Invariants in Pattern Recognition*. John Wiley & Sons, 2009.

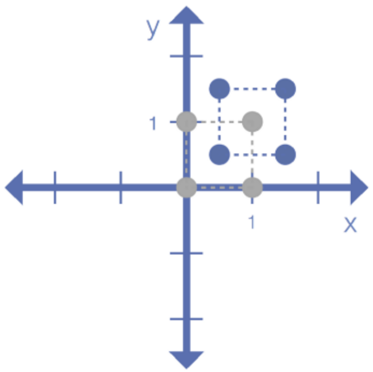
Geometric Transformations and Geometric Moments

- Let us consider the basic geometric transformations, including translation, rotation and scaling, which can be modeled as:

$$\mathbf{x}' = s\mathbf{R}_\alpha \mathbf{x} + \mathbf{t} \qquad \mathbf{R}_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

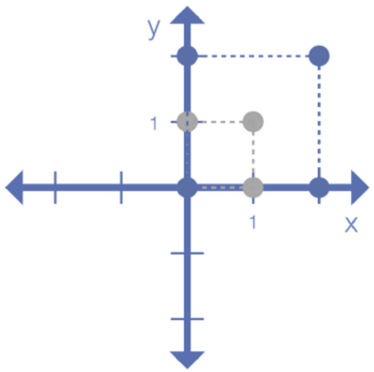
Translate

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



Scale

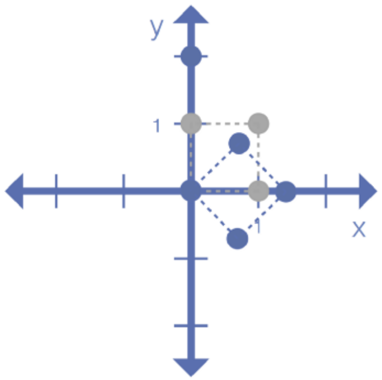
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotate

$$\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c = s = sin(45°)



- We can also define the so-called geometric moments with very simple basis functions:

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\,\mathrm{d}x\mathrm{d}y$$

- J. Flusser, B. Zitova, T. Suk. *Moments and Moment Invariants in Pattern Recognition*. John Wiley & Sons, 2009.

Translation and Scaling Invariants

- With the above definitions, translation invariants μ can be derived from the geometric moments:

$$\mu_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_c)^p (y - y_c)^q f(x, y)\,\mathrm{d}x\mathrm{d}y \qquad x_c = m_{10}/m_{00}, \quad y_c = m_{01}/m_{00}$$

- where (x_c, y_c) should be considered as the centroid of the image. The invariance is achieved by aligning the coordinate origin of the basis functions with the centroid.
- Let us further consider scaling invariants ν, which again can be derived from geometric moments, by normalizing the scaling factor on moments:

$$\mu'_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_c)^p (y - y_c)^q f(x/s, y/s)\,\mathrm{d}x\mathrm{d}y \qquad \nu_{pq} = \frac{\mu_{pq}}{\mu_{00}^{w}} \quad w = \frac{p+q}{2} + 1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^p (x - x_c)^p s^q (y - y_c)^q f(x, y) s^2\,\mathrm{d}x\mathrm{d}y = \boxed{s^{p+q+2}}\mu_{pq} \qquad \nu'_{pq} = \frac{\mu'_{pq}}{(\mu'_{00})^w} = \frac{s^{p+q+2}\mu_{pq}}{(s^2\mu_{00})^w} = \nu_{pq}$$

- J. Flusser, B. Zitova, T. Suk. *Moments and Moment Invariants in Pattern Recognition*. John Wiley & Sons, 2009.

Rotation Invariants by Hu and Hilbert

- Are rotation invariants ϕ equally derivable from geometric moments? Yes, **Hu** gives 7 invariants based on **Hilbert's algebraic invariants**, which seems very complex. But it makes sense, due to the nonlinear action of the rotations on x and y.

$$\phi_1 = m_{20} + m_{02},$$

$$\phi_2 = (m_{20} - m_{02})^2 + 4m_{11}^2,$$

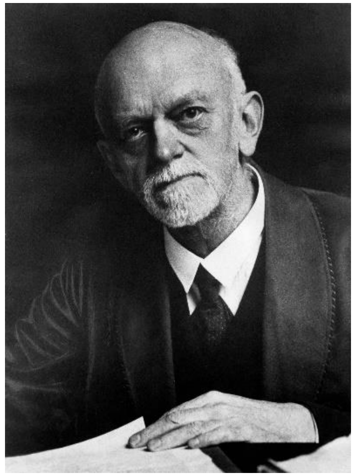
$$\phi_3 = (m_{30} - 3m_{12})^2 + (3m_{21} - m_{03})^2,$$

$$\phi_4 = (m_{30} + m_{12})^2 + (m_{21} + m_{03})^2,$$

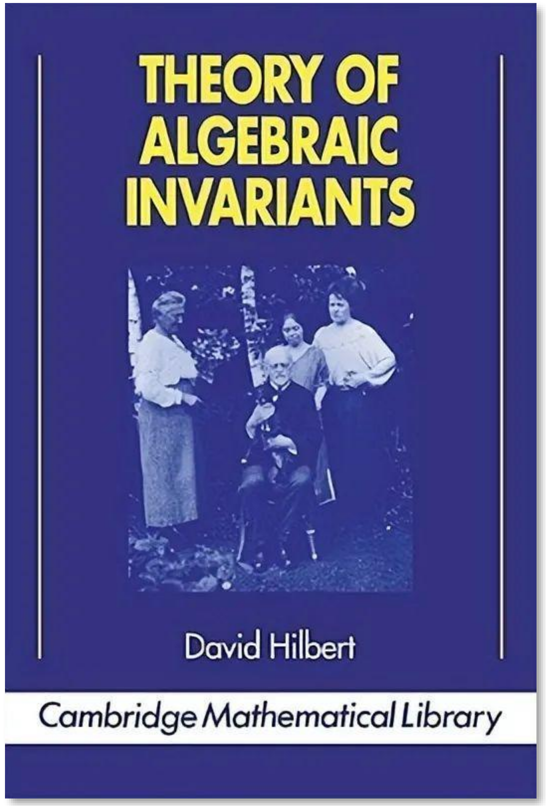
$$\begin{aligned}\phi_5 = {} & (m_{30} - 3m_{12})(m_{30} + m_{12})((m_{30} + m_{12})^2 - 3(m_{21} + m_{03})^2) \\ & + (3m_{21} - m_{03})(m_{21} + m_{03})(3(m_{30} + m_{12})^2 - (m_{21} + m_{03})^2),\end{aligned}$$

$$\begin{aligned}\phi_6 = {} & (m_{20} - m_{02})((m_{30} + m_{12})^2 - (m_{21} + m_{03})^2) \\ & + 4m_{11}(m_{30} + m_{12})(m_{21} + m_{03}),\end{aligned}$$

$$\begin{aligned}\phi_7 = {} & (3m_{21} - m_{03})(m_{30} + m_{12})((m_{30} + m_{12})^2 - 3(m_{21} + m_{03})^2) \\ & - (m_{30} - 3m_{12})(m_{21} + m_{03})(3(m_{30} + m_{12})^2 - (m_{21} + m_{03})^2).\end{aligned}$$



D. Hilbert, 1897
Algebraic Invariants



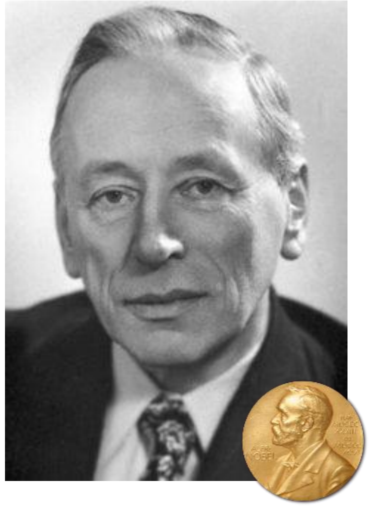
- MK Hu. Visual pattern recognition by moment invariants. *TIT*, 1962.

Rotation Invariants by Zernike

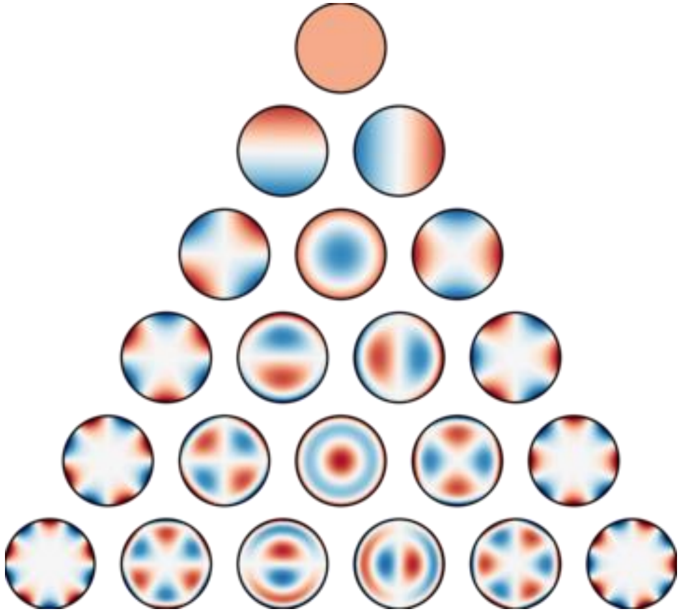
- Can rotation invariants be simpler? Let us define the basis functions in polar coordinates, where rotations are more easily managed, by the Fourier translation invariance in the angular form.
- In this respect, Zernike polynomials are typical — they are complete orthogonal bases on the unit circle and easily realize rotation invariance, from **Zernike's optical research**.

$$C_{pq} = \int_0^{\infty}\int_0^{2\pi} R_{pq}(r)e^{i\xi(p,q)\theta}f(r,\theta)r\mathrm{d}\theta\mathrm{d}r$$

$$\begin{aligned} C'_{pq} &= \int_0^{\infty}\int_0^{2\pi} R_{pq}(r)e^{i\xi(p,q)\theta}f(r,\theta+\alpha)r\mathrm{d}\theta\mathrm{d}r \\ &= \int_0^{\infty}\int_0^{2\pi} R_{pq}(r)e^{i\xi(p,q)(\theta-\alpha)}f(r,\theta)r\mathrm{d}\theta\mathrm{d}r \\ &= \boxed{e^{-i\xi(p,q)\alpha}}C_{pq}. \end{aligned}$$



F. Zernike, 1934
Zernike Polynomials



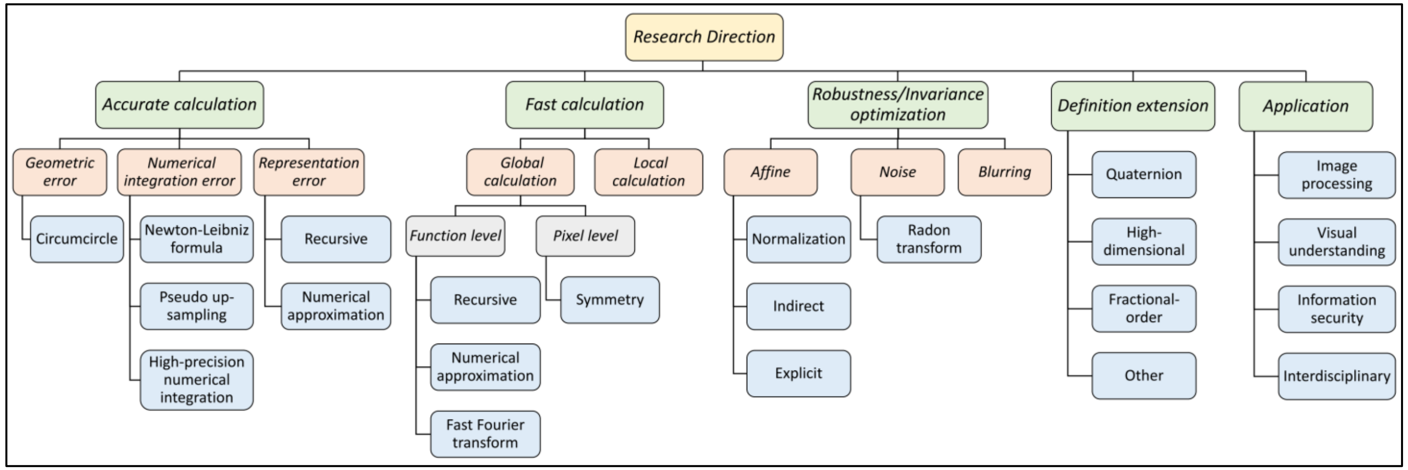
- A Khotanzad, YH Hong. Invariant image recognition by Zernike moments. *TPAMI*, 1990.

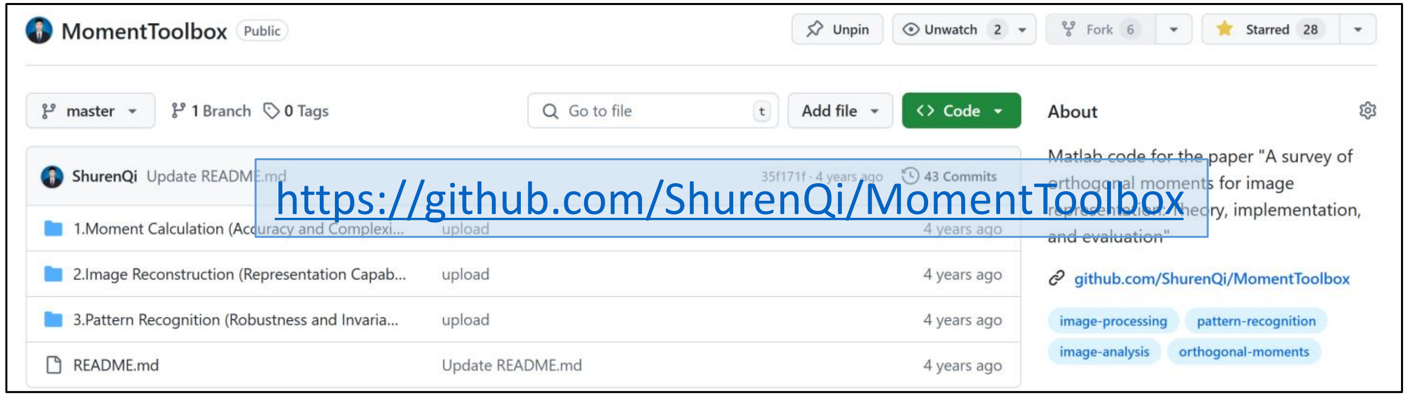
Our Contributions

	Moment Invariants	Multiscale and Wavelet	CNN to Geometry Deep Learning
	2000s	*2010s*	*2020s*
	Flusser	Lowe Lindeberg Mallat…	LeCun Bronstein…
Assumption	Global	Local	Hierarchical
Invari./Discri.	Invariance	Mid?	Discriminability
Robust./Interp./Effic.	Good	Mid?	Not Good
Our Contributions	CSUR’23 TPAMI’24 TIFS review	TPAMI’23 USENIX Security’25 TIP’25	ICCV’25 TIP’25

Refining Global Invariants

- We give papers on the practical aspects of moments for refining global invariants, covering numerical analyses, software implementations, benchmark evaluations, and recent advances.





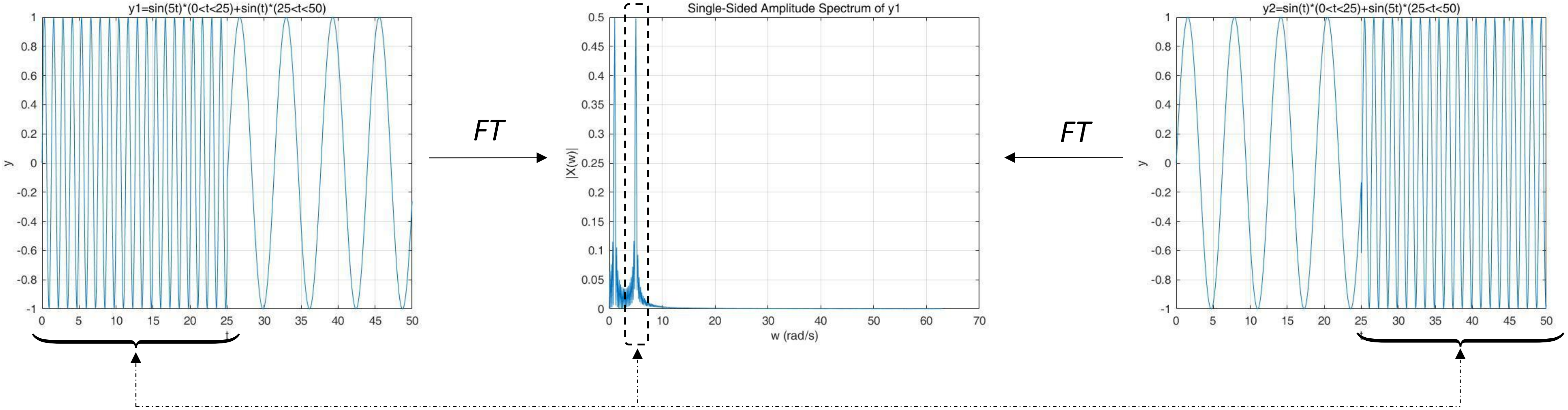
https://github.com/ShurenQi/MomentToolbox

- S. Qi, Y. Zhang, C. Wang, et al. A Survey of Orthogonal Moments for Image Representation: Theory, Implementation, and Evaluation. *ACM Computing Surveys (CSUR)*, 2023, 55(1): 1-35.
- S. Qi, Y. Zhang, C. Wang, et al. Representing Noisy Image Without Denoising. *IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI)*, 2024, 46(10): 6713 - 6730

From Global To Local

Why We Need Local Representations

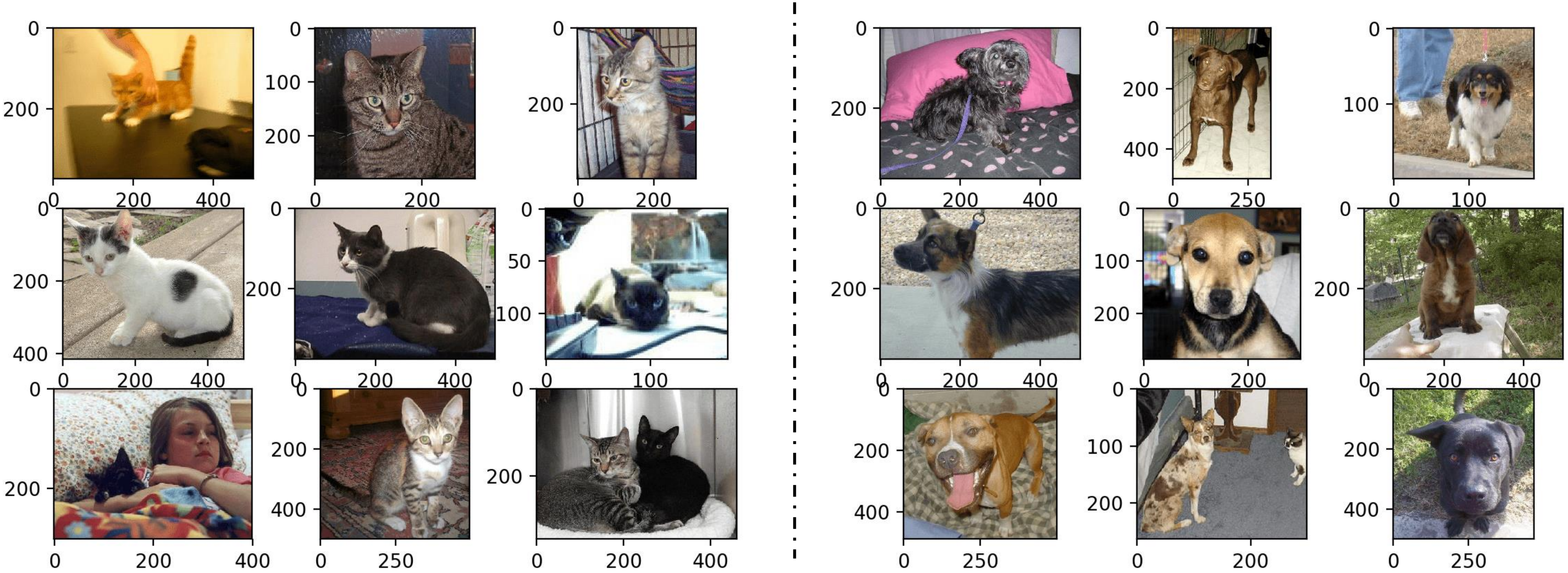
- Fourier transform-like global representations are typically (under)-complete and are just designed for low-level processing, struggling to express high-level semantics with over-completeness.
- As a toy example, the Fourier transform cannot even distinguish the order in which the two signals appear.



- AV Oppenheim, JS Lim. The importance of phase in signals. *Proceedings of the IEEE*, 1981.

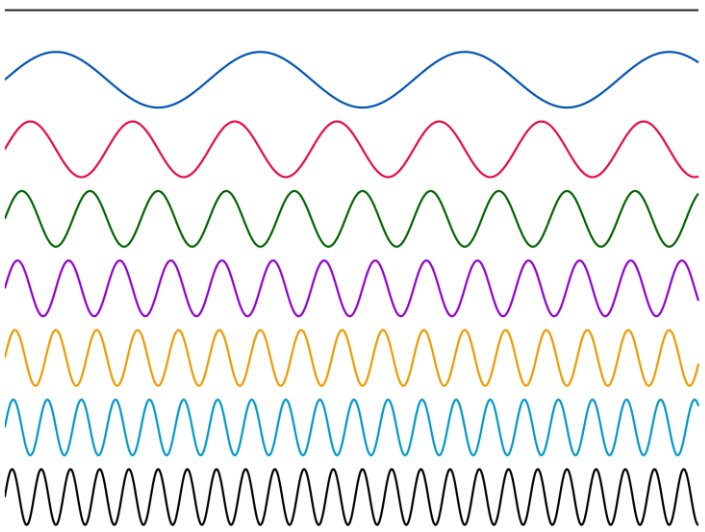
Why We Need Local Representations

- Under realistic considerations, there are too many tasks concerned with local semantic properties — recognition and classification (distinguish images of cats and dogs), where global representations are likely unable to provide enough information to support discriminability.



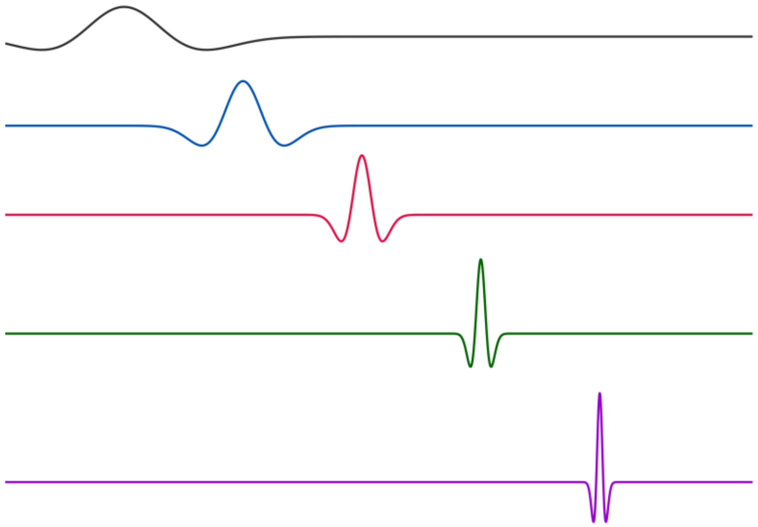
Local Representations: Wavelet Transform

- Different from Fourier, basis functions of **Wavelet Transform** are local and multi-scale.



$$\hat{x}(\xi) = \int_{-\infty}^{+\infty} x(u)e^{-\mathrm{i}\xi u}\mathrm{d}u$$

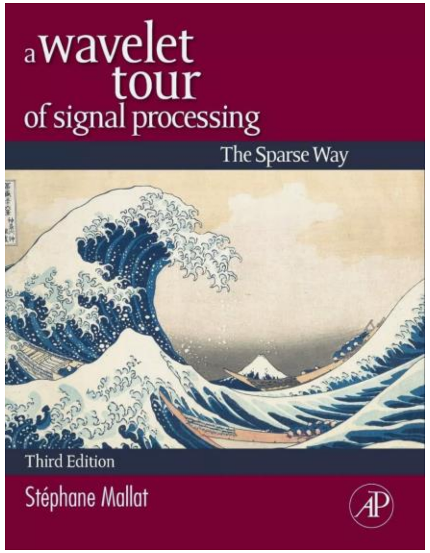
Fourier



$$(W_\psi x)(u,\xi) = \xi^{-1/2}\int_{-\infty}^{+\infty} \psi\boxed{\left(\frac{v-u}{\xi}\right)} x(v)\mathrm{d}v$$

Wavelets

S. Mallat, 1999
Wavelets

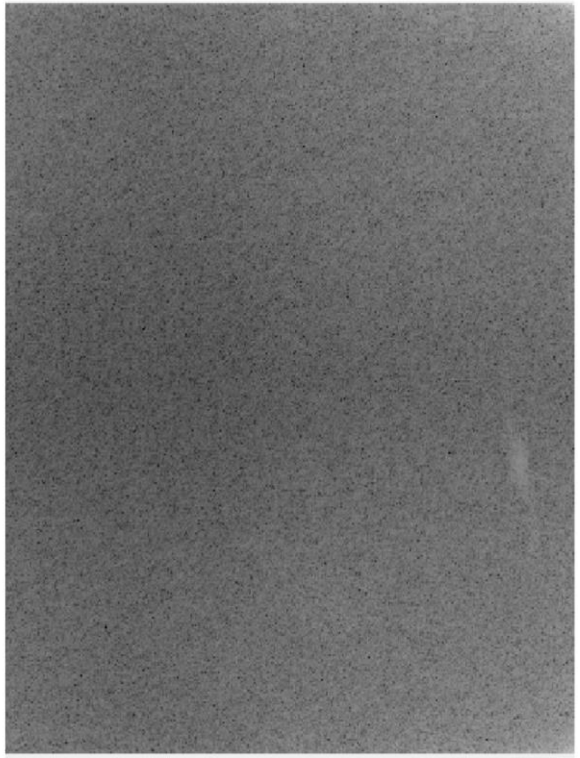


- S Mallat. *A Wavelet Tour of Signal Processing*. Elsevier, 1999.

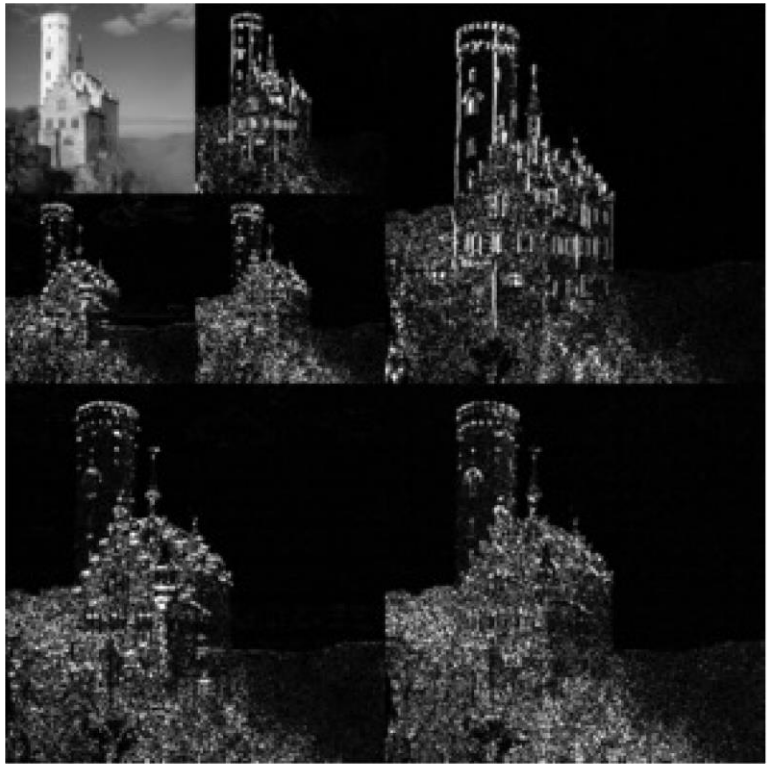
Local Representations: Wavelet Transform

- Wavelet transform can capture local information, with better discriminative properties — time-frequency discriminability and over-completeness.

Original Image



Fourier Representations

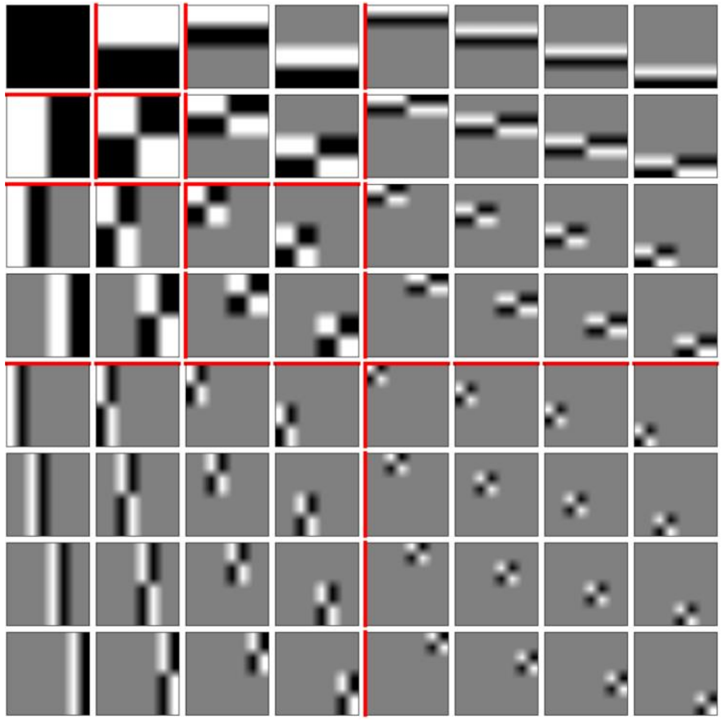


Wavelet Representations

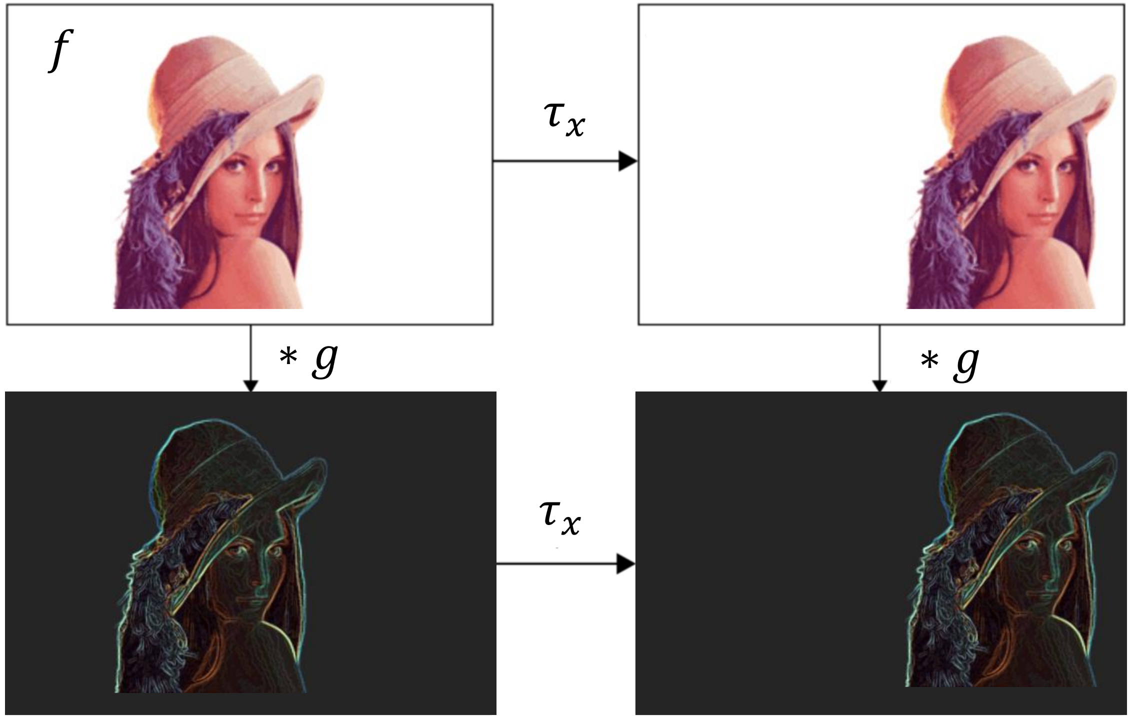
So, How About Invariance?

Translation Equivariance of Wavelet Transform

- The wavelet basis functions define convolution operators g — the wavelet transform of an image f means the convolution of f and g. Therefore, the wavelet transform has a translation equivariance with the convolution.



Wavelet Convolutional Operators g

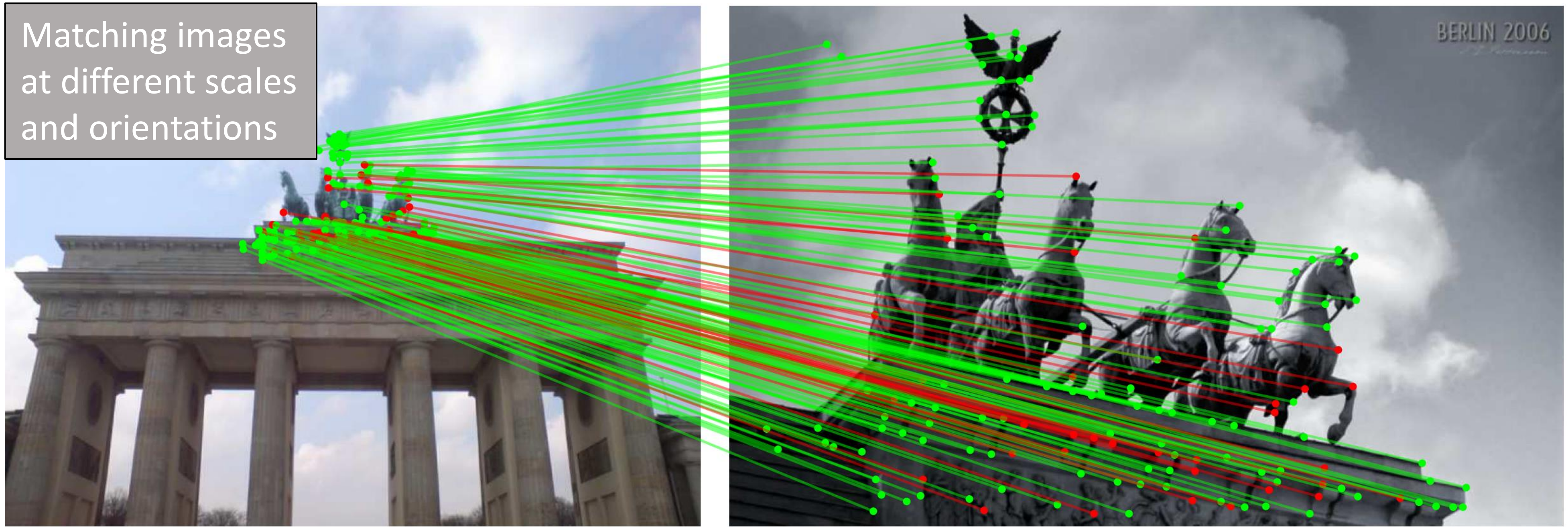


$$(\tau_x f) * g = \tau_x (f * g)$$

Can Local Invariance Be Generalized To Other Geometric Transformations?

Local Representations: SIFT

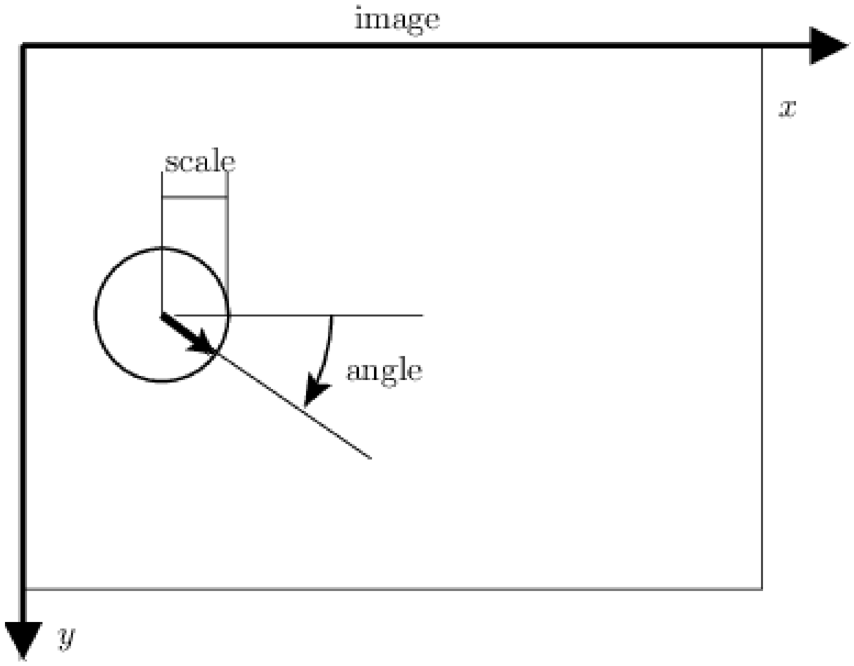
- The local and multiscale concepts of the wavelet transform were followed by later local representations.
- For example, the well-known **Scale-Invariant Feature Transform (SIFT)** aims at the local invariance of rotation and scaling in multiscale spaces.



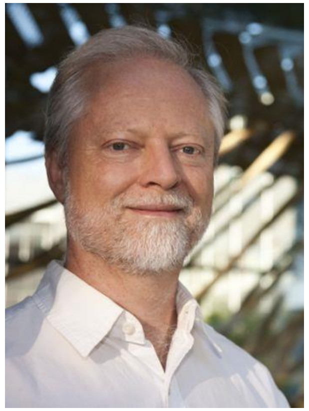
- DG Lowe. Distinctive image features from scale-invariant keypoints. *IJCV*, 2004.

Local Representations: SIFT

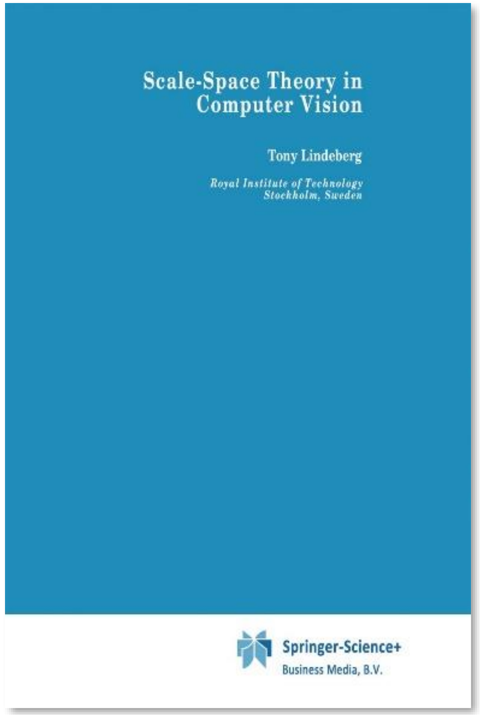
- SIFT describes local regions that have their own scale and orientation, with the **scale space theory** as a foundation.
- Here, once the scale and orientation of the regions can be evaluated stably, then invariant features can be constructed by normalizing the scale and orientation.



T. Lindeberg, 1993
Scale Space Theory



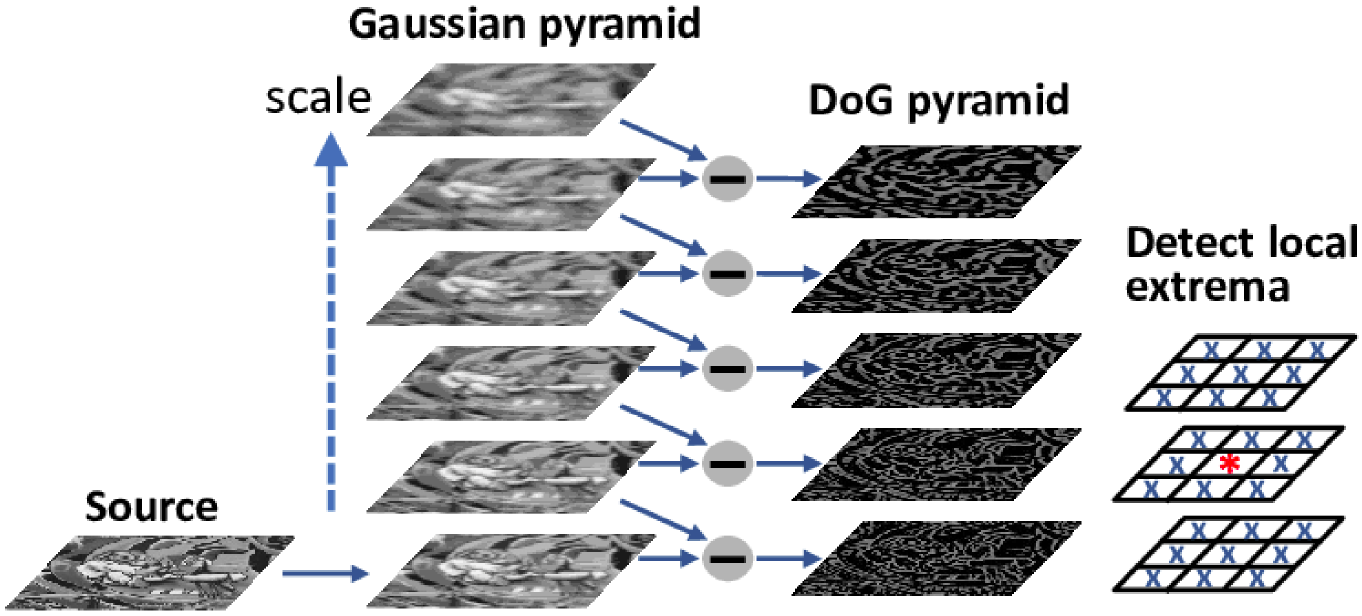
D. Lowe, 1999
SIFT



- T Lindeberg. *Scale-space Theory in Computer Vision*. Springer Science & Business Media, 1993.

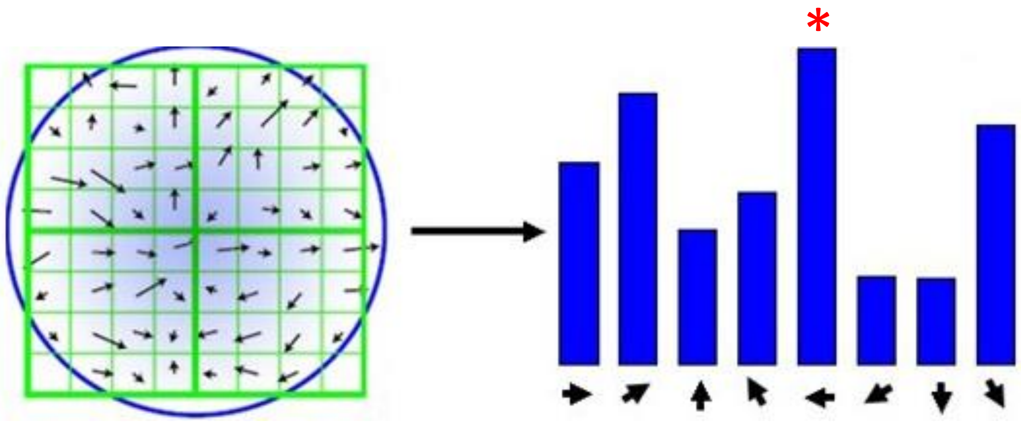
Local Representations: SIFT

- SIFT has two main components: detector and descriptor.
- The detector is responsible for finding the interest point with evaluated scale to achieve scaling invariance. The descriptor is responsible for describing the interest point with evaluated orientation to further achieve rotation invariance.



Detector

Scale is evaluated by finding the extreme in the scale space



Descriptor

Orientation is evaluated by computing the histogram of gradients

From Sparse To Dense

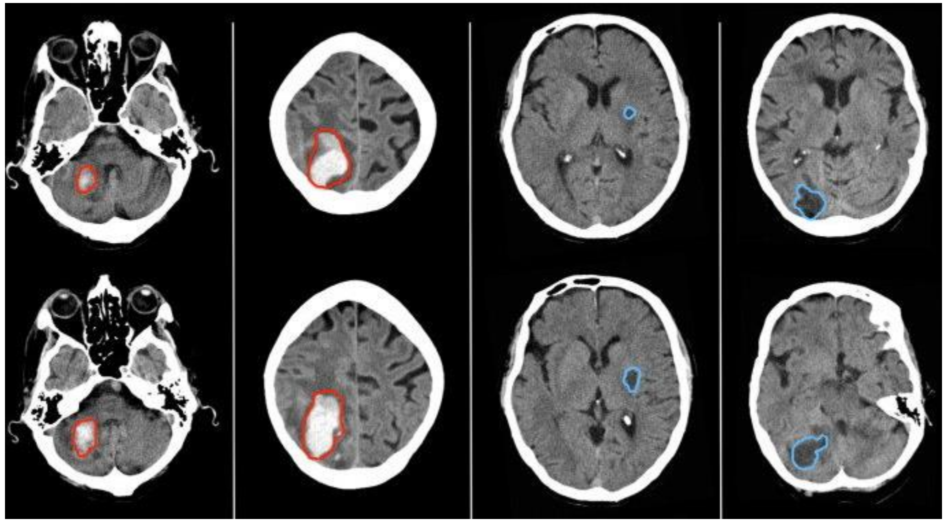
Why We Need Dense Representations

- SIFT-like interest points are sparse in the image and are designed to focus only on the main subject (ignoring all other regions).

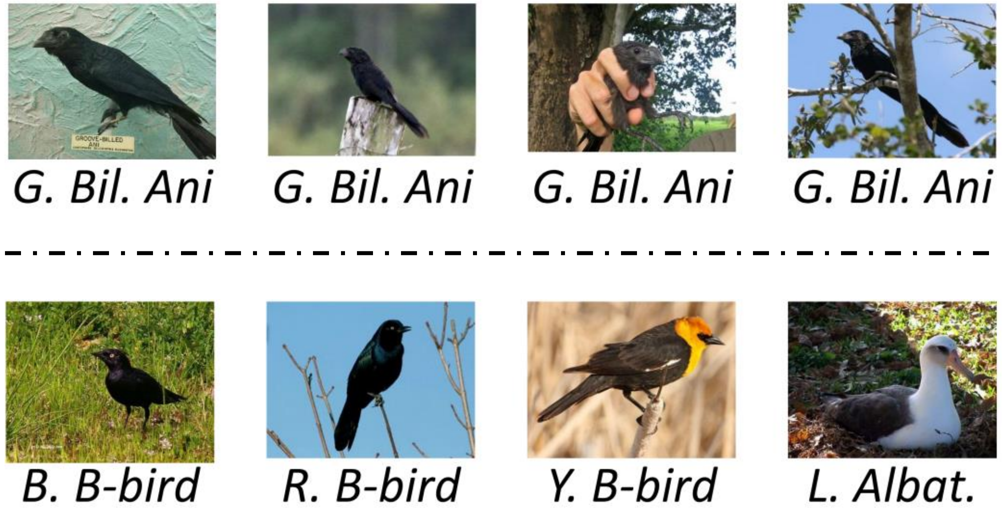
- A Iscen, G Tolias, PH Gosselin, et al. A comparison of dense region detectors for image search and fine-grained classification. *TIP*, 2015.

Why We Need Dense Representations

- Under realistic considerations, there are too many tasks concerned with dense semantic properties — detection/localization (detect lesions in CT images), fine-grained classification (distinguish large-scale bird images), where sparse points are likely to miss important local information.



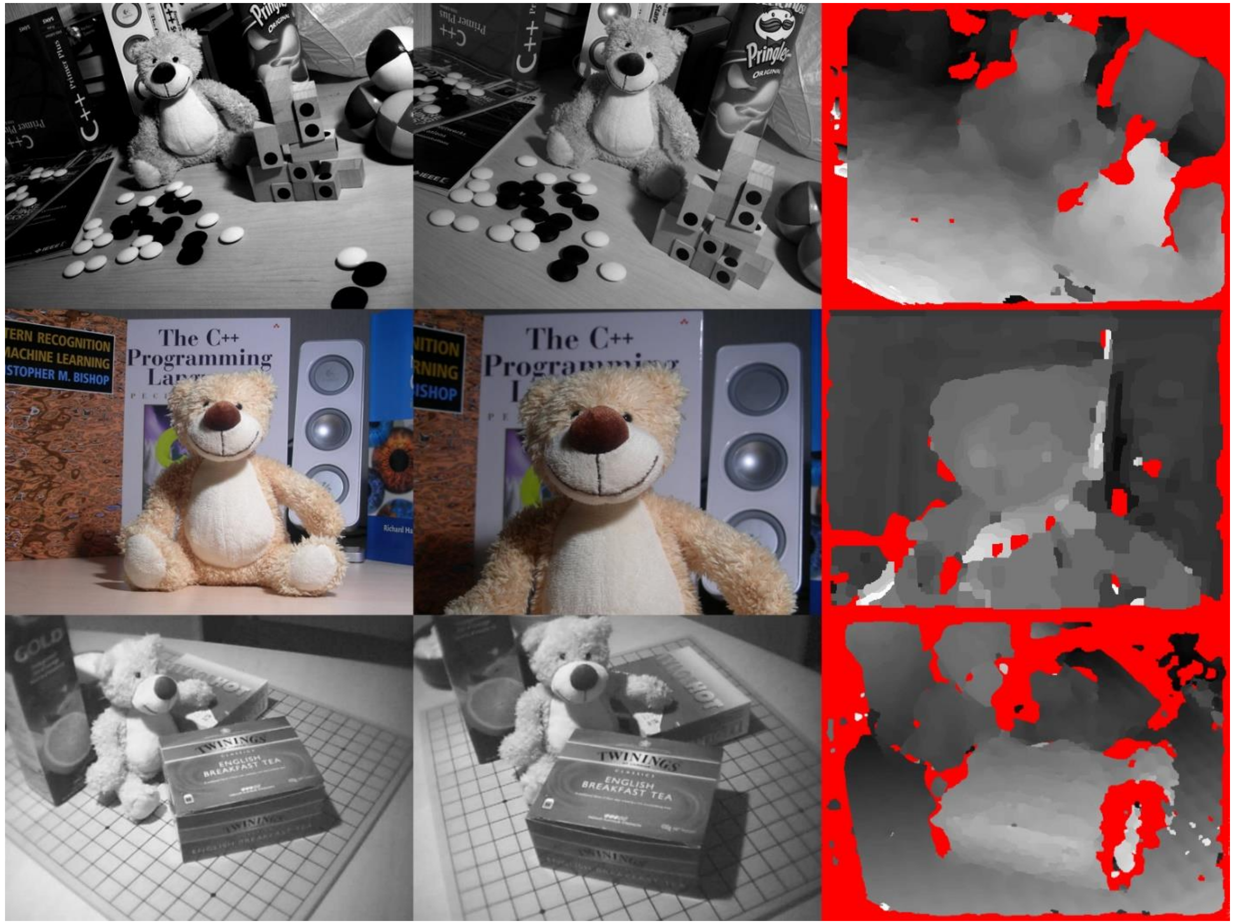
Detection/Localization



Fine-grained Classification

Local Representations: DAISY

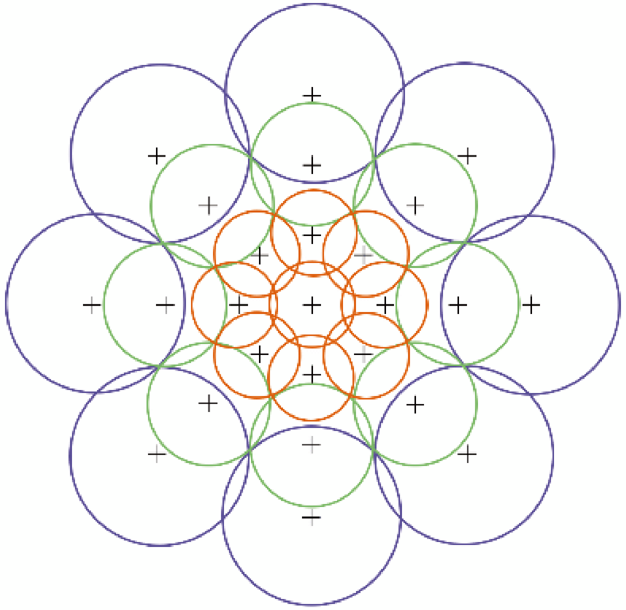
- **DAISY** aims to extend SIFT from sparse to dense, achieving local invariance of rotation and scaling for each pixel position.



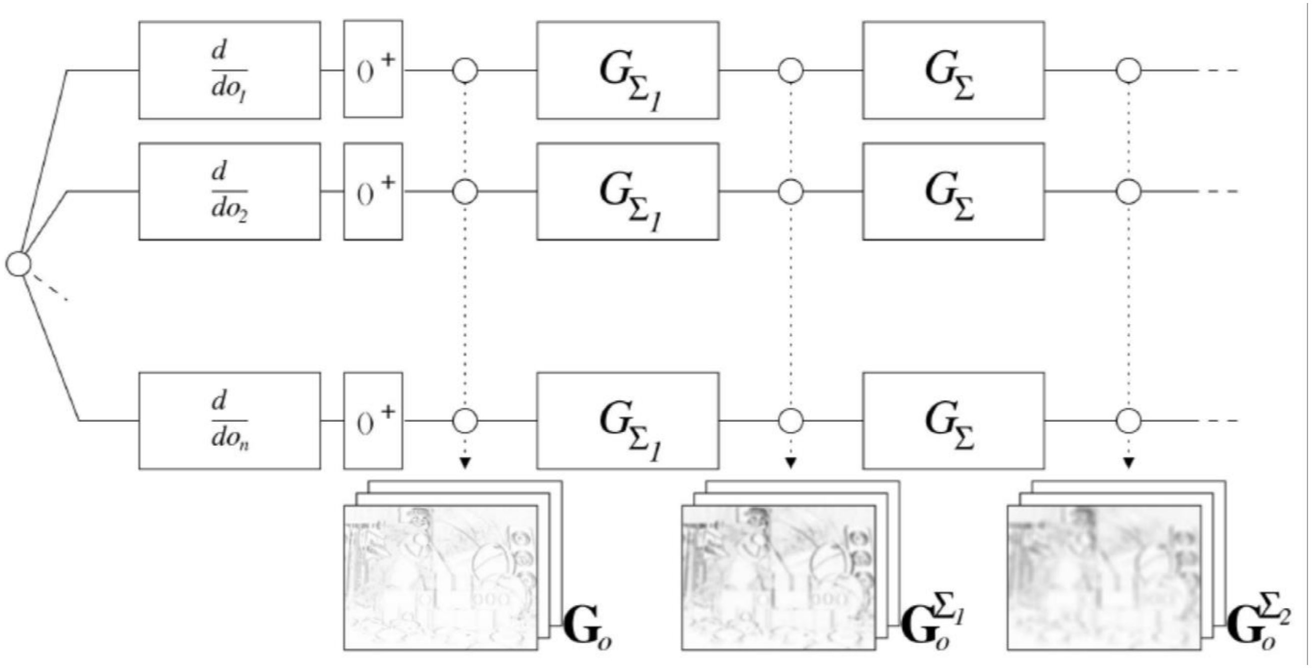
- E Tola, V Lepetit, P Fua. Daisy: An efficient dense descriptor applied to wide-baseline stereo. *TPAMI*, 2009.

Local Representations: DAISY

- The main difficulty is that the complex operations of SIFT in scale and orientation evaluation cannot be performed directly for dense positions, due to high complexity.
- Therefore, DAISY introduces a series of simplified designs for scale and orientation, but at the same time invariance is reduced.



DAISY Descriptor



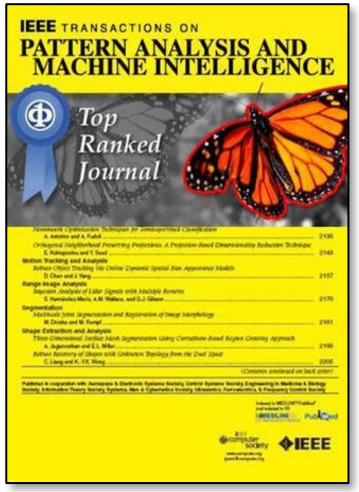
Simplified Designs for Scale and Orientation

Our Contributions

	Moment Invariants	Multiscale and Wavelet	CNN to Geometry Deep Learning
	2000s	*2010s*	*2020s*
	Flusser	Lowe Lindeberg Mallat…	LeCun Bronstein…
Assumption	Global	Local	Hierarchical
Invari./Discri.	Invariance	Mid?	Discriminability
Robust./Interp./Effic.	Good	Mid?	Not Good
Our Contributions	CSUR'23 TPAMI'24 TIFS review	TPAMI'23 USENIX Security'25 TIP'25	ICCV'25 TIP'25

Designing Local Invariants

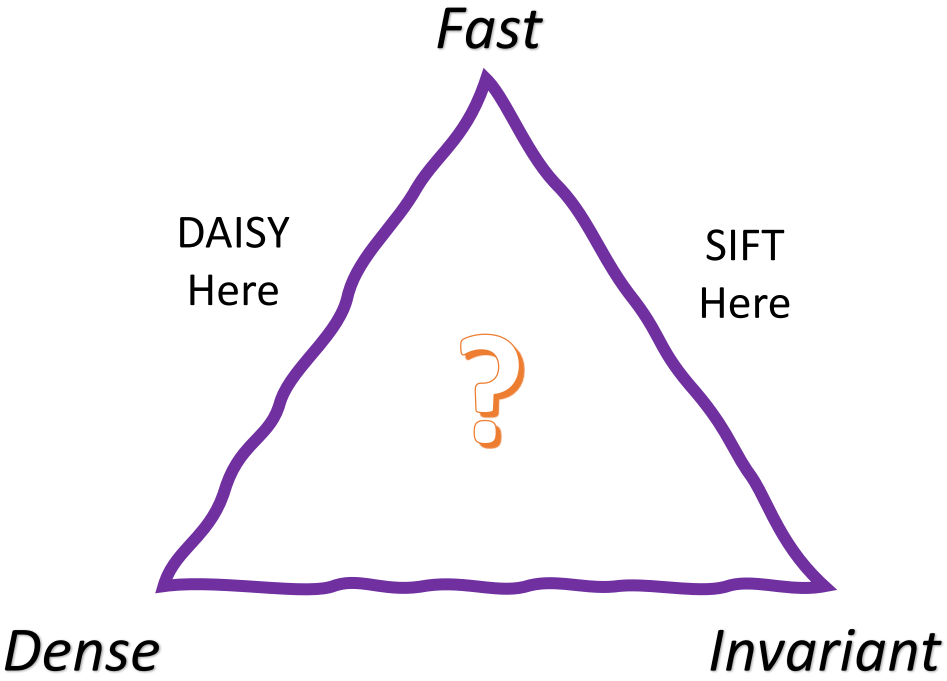
- Reviewing the above local invariants, one can note a gap: SIFT is fast and invariant, but not suitable for dense tasks; DAISY is fast and dense, but largely compresses invariance.
- We tried to define truly dense invariants while being fast enough. We achieved this goal by exploring the potential of **classical moment invariants**.



A Principled Design of Image Representation: Towards Forensic Tasks

Shuren Qi, Yushu Zhang, Member, IEEE, Chao Wang, Jiantao Zhou, Senior Member, IEEE, and Xiaochun Cao, Senior Member, IEEE

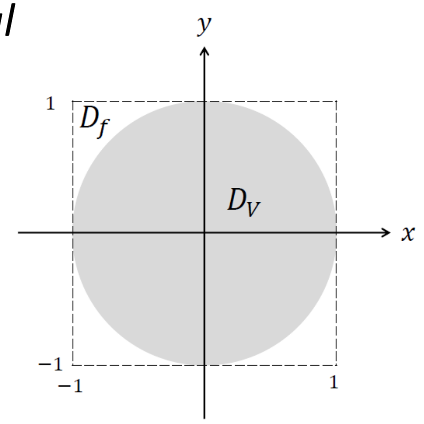
- S. Qi, Y. Zhang, C. Wang, et al. A Principled Design of Image Representation: Towards Forensic Tasks. *IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI)*, 2023, 45(5): 5337 - 5354



Moments: From Global to Local

- First, we extend the definition of classical moments from the global to the local with scale space. Here, local coordinate system (x', y') is a translated and scaled version of the global coordinate system (x, y), with translation offset (u, v) and scale factor w.
- Two interesting properties: generic nature and local representation capability.

Global



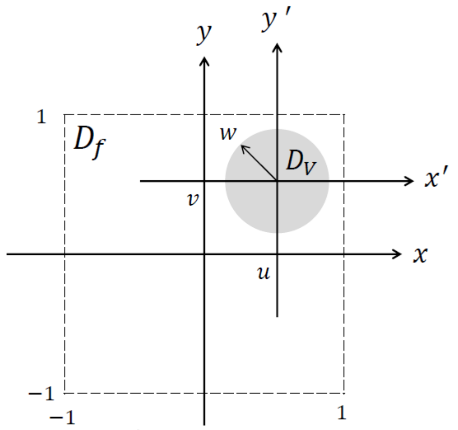
$$\langle f, V_{nm} \rangle = \iint_D R_n^*(r) A_m^*(\theta) f(r,\theta) r\,dr\,d\theta$$

Our Transformations

$$(x', y') = \frac{(x,y) - (u,v)}{w}$$

$$\begin{cases} r' = \sqrt{(x')^2 + (y')^2} = \frac{1}{w}\sqrt{(x-u)^2 + (y-v)^2} \\ \theta' = \arctan(\frac{y'}{x'}) = \arctan(\frac{y-v}{x-u}) \end{cases}$$

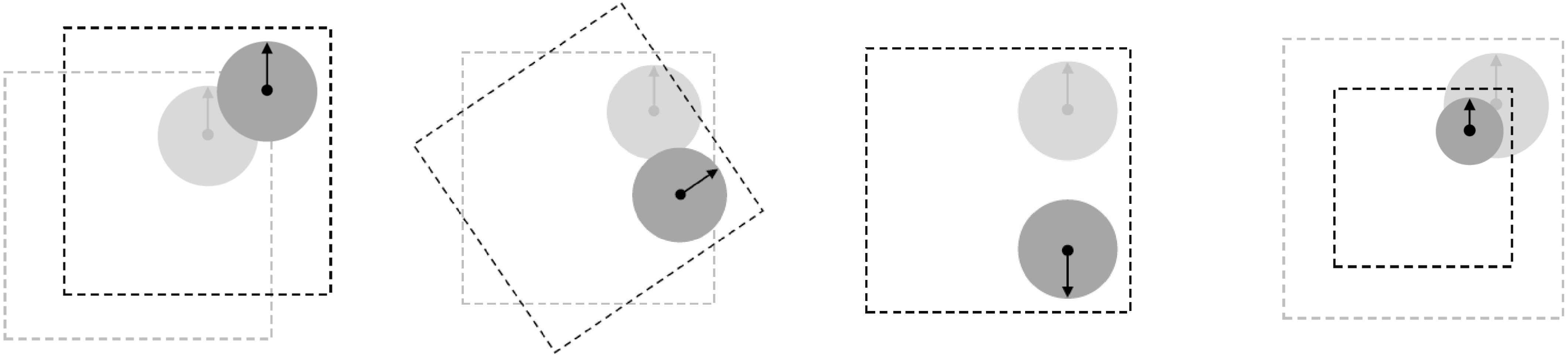
Local



$$\langle f, V_{nm}^{uvw} \rangle = \iint_D \underbrace{R_n^*(\overbrace{\frac{\sqrt{(x-u)^2 + (y-v)^2}}{w}}^{r'}) A_m^*(\overbrace{\arctan(\frac{y-u}{x-v})}^{\theta'})}_{(V_{nm}^{uvw}(x,y))^*} f(x,y)\,dx\,dy$$

Moment Invariants: From Global to Local

- Then, we found the symmetry properties of the local definition for several geometric transformations.
- Therefore, rotation and flipping invariants can be obtained by taking the absolute values; translation and scaling invariants can be obtained by pooling over the $(u, v)/w$.



$$\langle f(x+\Delta x, y+\Delta y), V_{nm}^{uvw}(x,y)\rangle = \left\langle f(x,y), V_{nm}^{(u+\Delta x)(v+\Delta y)w}(x,y)\right\rangle$$

Translation Equivariance
w.r.t. (u, v)

$$\langle f(r,\theta+\phi), V_{nm}^{uvw}(r',\theta')\rangle = \langle f(r,\theta), V_{nm}^{uvw}(r',\theta')\rangle\, A_m^*(-\phi)$$

Rotation Invariance
w.r.t. absolute values

$$\langle f(r,-\theta), V_{nm}^{uvw}(r',\theta')\rangle = (\langle f(r,\theta), V_{nm}^{uvw}(r',\theta')\rangle)^*$$

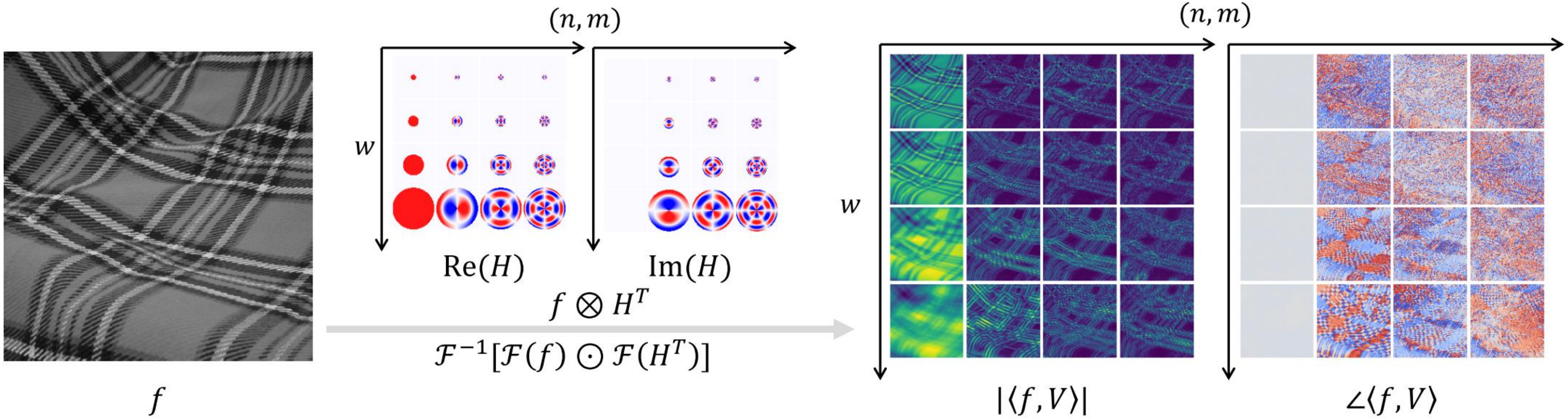
Flipping Invariance
w.r.t. absolute values

$$\langle f(sx,sy), V_{nm}^{uvw}(x,y)\rangle = \left\langle f(x,y), V_{nm}^{uv(ws)}(x,y)\right\rangle$$

Scaling Covariance
w.r.t. w

Fast Implementation

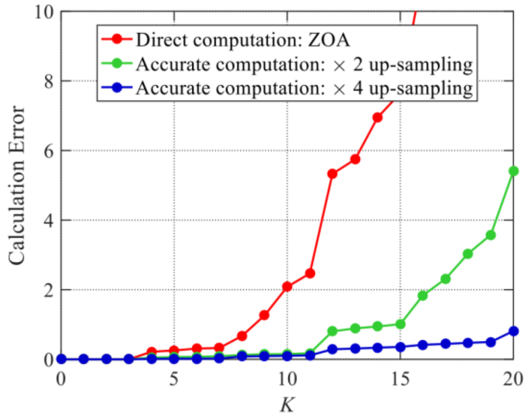
- Finally, we give a fast implementation by the convolution theorem.

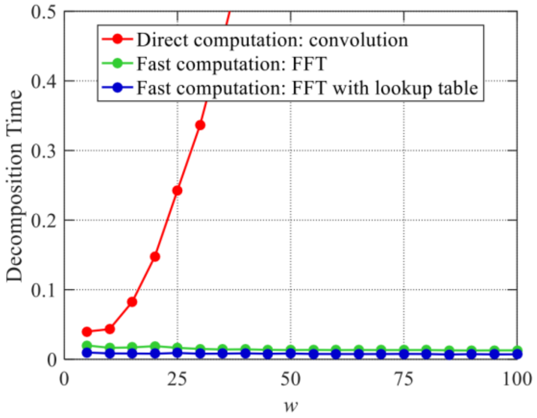


$$\mathcal{O}({w_{\max}}^2 \#_{uv} \#_w) \quad \text{VS} \quad \mathcal{O}(\#_w \#_{uv} \log \#_{uv})$$

$$\Downarrow$$

$${w_{\max}}^2 \quad \text{VS} \quad \log(\#_{uv})$$





Tutorial Outline

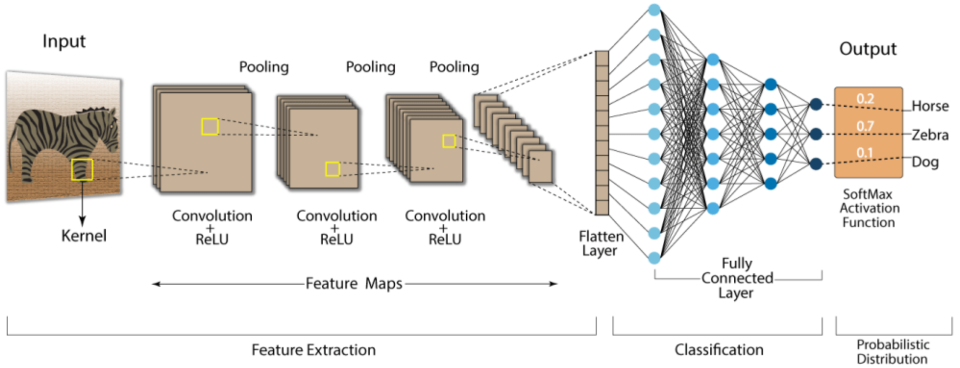
- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

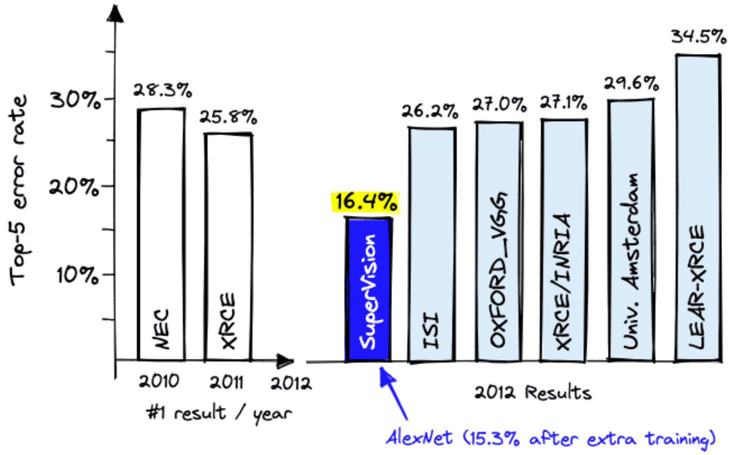
A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

From Knowledge Driven To Data Driven

Invariance in The Early Era of Deep Learning

- Knowledge Driven: Despite decades of research, these hand-crafted representations still fail to provide sufficient discriminability for large-scale tasks, especially in the discrimination of real-world semantic content.
- Data Driven: As we enter the early era of deep learning, convolutional neural networks achieve strong discriminative power for large-scale tasks, known as *ImageNet moment*.





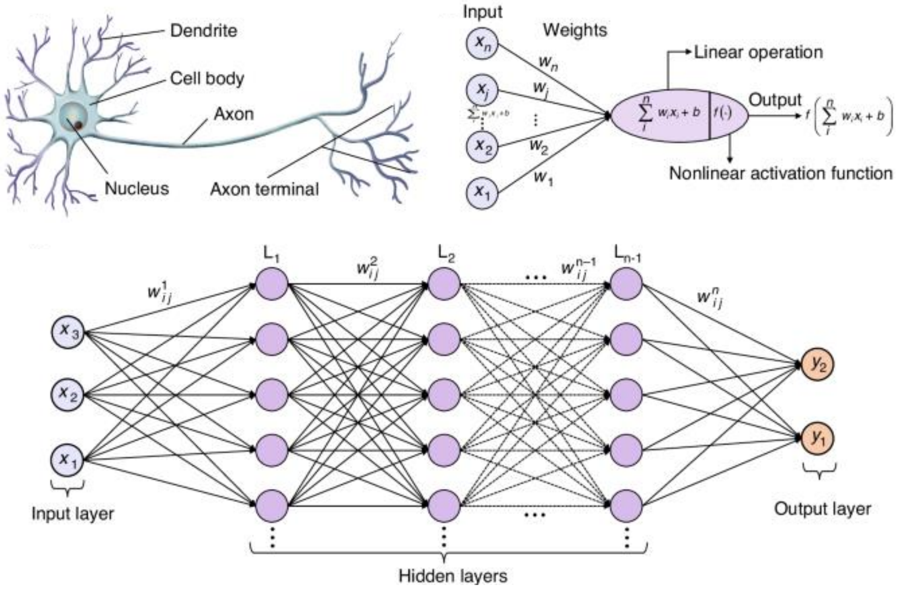
The Success of Deep Learning
AlexNet Wins ILSVRC 2012 Competition

A. Krizhevsky, 2012
AlexNet

- A Krizhevsky, I Sutskever, GE Hinton. ImageNet classification with deep convolutional neural networks. *NIPS* , 2012.

A Huge Span of Time

- Neural networks were proposed quite early, dating back to 1950s for the perceptron; but it was not until AlexNet in 2012 that the remarkable achievement was realized.
- What is the missing key?




Vol. VI, No. 2, Summer 1958

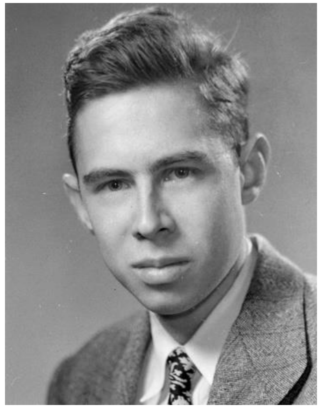
research trends

CORNELL AERONAUTICAL LABORATORY, INC., BUFFALO 21, NEW YORK

The Design of an Intelligent AUTOMATON

by FRANK ROSENBLATT

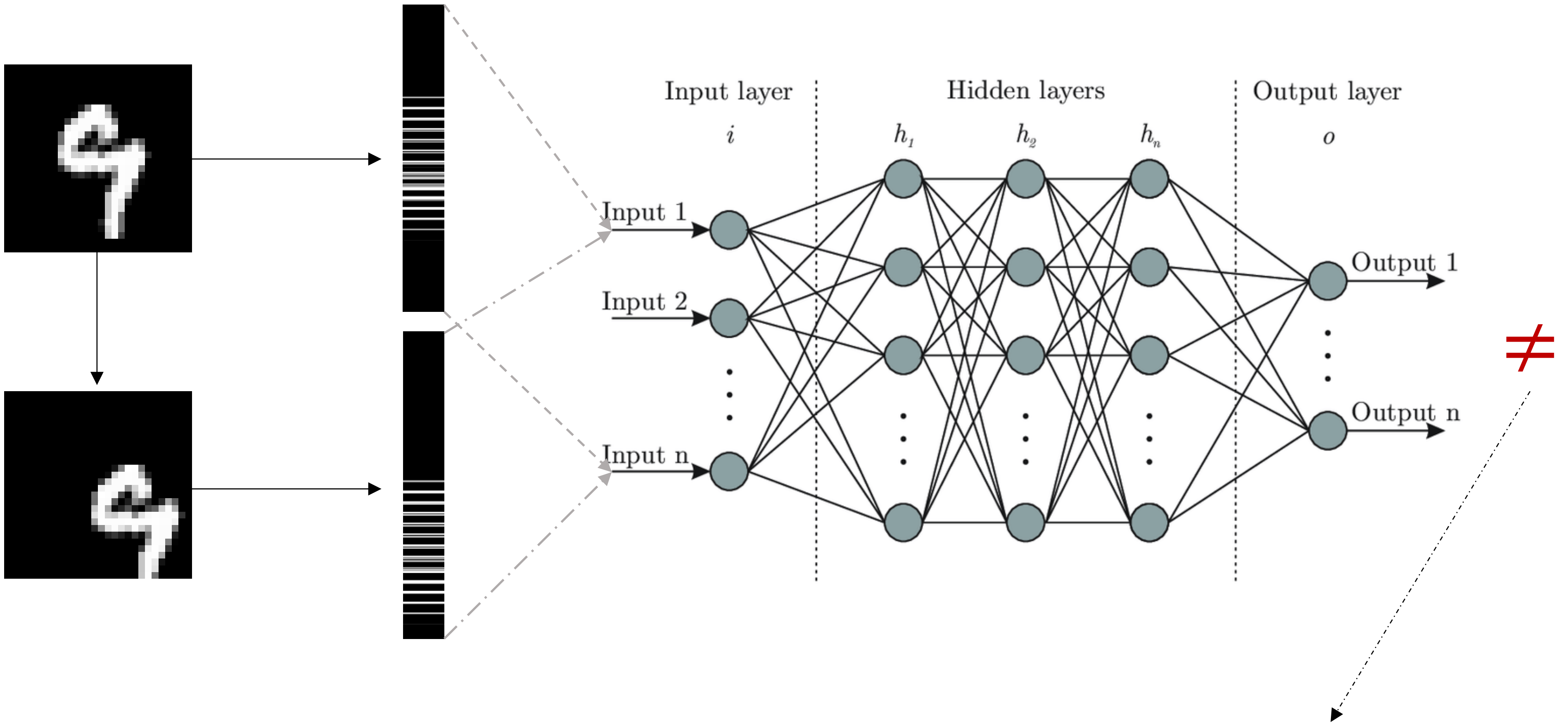
Introducing the perceptron — A machine which senses, recognizes, remembers, and responds like the human mind.



F. Rosenblatt, 1958
Perceptron

- F Rosenblatt. The perceptron: a probabilistic model for information storage and organization in the brain. *Psychological Review*, 1958.

Translations on Neural Networks



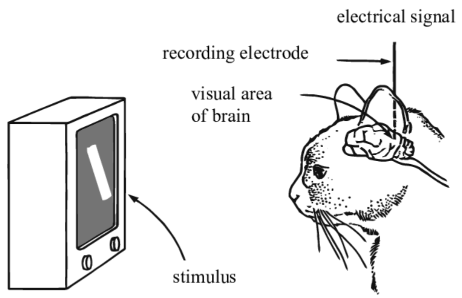
“The response of [Perceptrons] was severely affected by the shift in position […] of the input patterns. Hence, their ability for pattern recognition was not so high.” — Fukushima

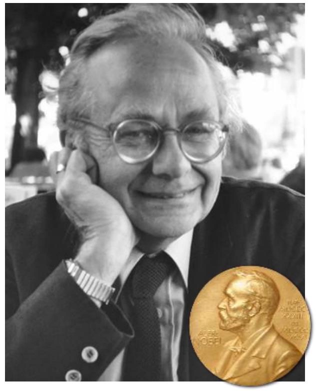
- K Fukushima, S Miyake. Neocognitron: A new algorithm for pattern recognition tolerant of deformations and shifts in position. *Pattern Recognition*, 1982.

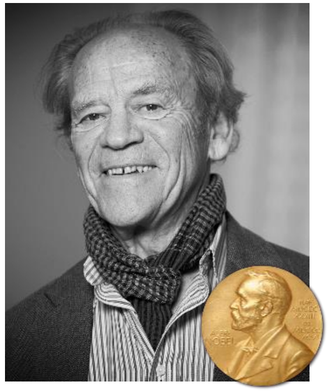
Convolution And Translation Equivariance

Convolutional Neural Networks

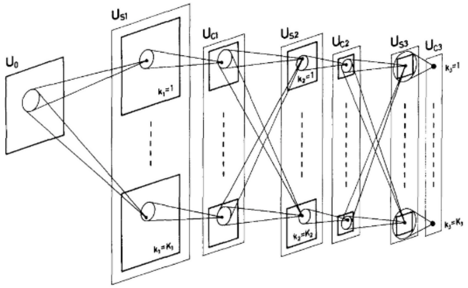
- Convolution with its Translation Equivariance (see also Wavelet Transform) are the key to enabling neural networks successful in visual tasks.
  - First, local structures was discovered in the biological vision by Hubel and Wiesel.
  - Then, convolution was introduced into neural networks by Fukushima.
  - Finally, such networks were equipped with learnability and backpropagation by LeCun.
- Invariance still plays an important role, even in the rise of the learning paradigm.







D. Hubel & T. Wiesel, 1959
Structure of Visual Cortex

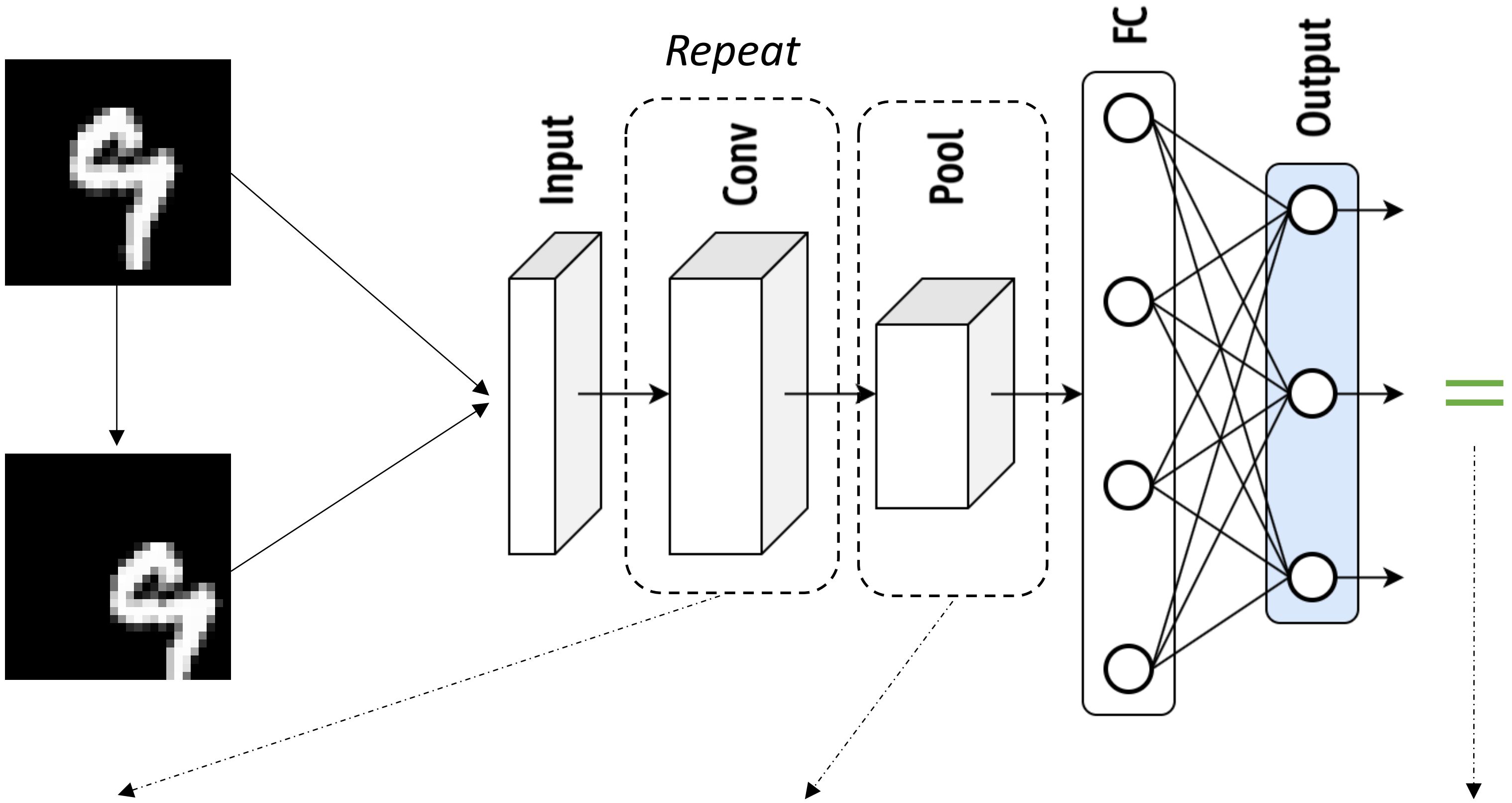


K. Fukushima, 1982
Neocognitron

Y. LeCun, 1989
LeNet

- Y LeCun, B Boser, J Denker, et al. Handwritten digit recognition with a back-propagation network. *NIPS*, 1989.

Translation Equi/In-variance of Convolutional Neural Networks



Translation Equivariance of Convolution	*Translation Invariance of Global Pooling*	*Hierarchical Invariance of Classification to Translation*
$(\tau_x f) * g = \tau_x (f * g)$	$\mathrm{pool}(\tau_x f) = \mathrm{pool}(f)$	$\mathrm{C}(\tau_x f) = \mathrm{C}(f)$

Tutorial Outline

- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

Time for a Break!!

Tutorial Outline

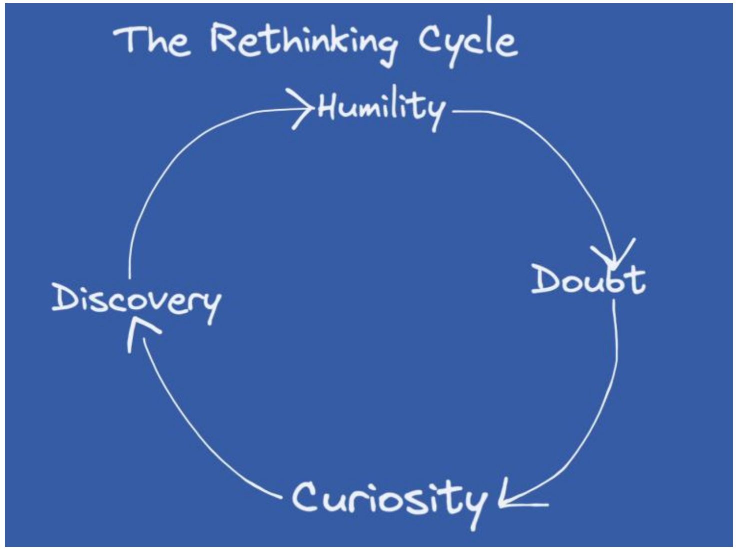
- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

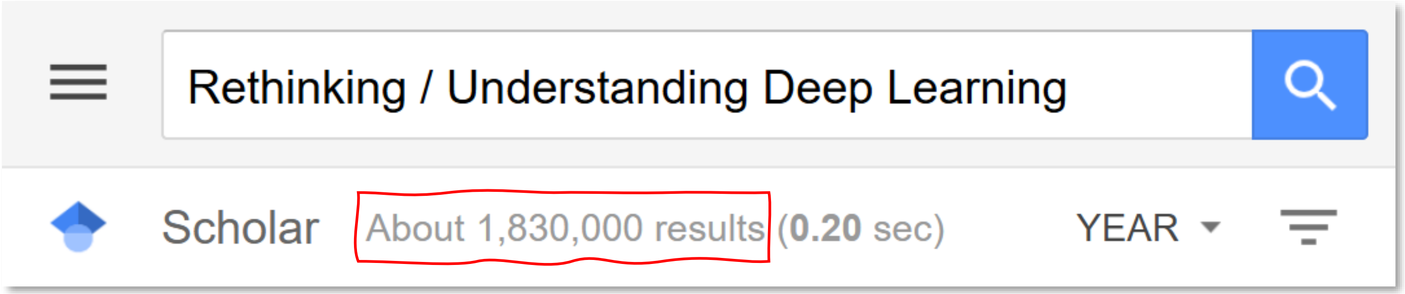
A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

Why Rethink Deep Learning?
Realistic Needs And Theoretical Extensions

Invariance in The Era of Rethinking Deep Learning

- Realistic Needs: Bottlenecks in robustness, interpretability, and efficiency.
- Theory Extensions: A unified theory perspective to avoid getting into endless experimental designs.

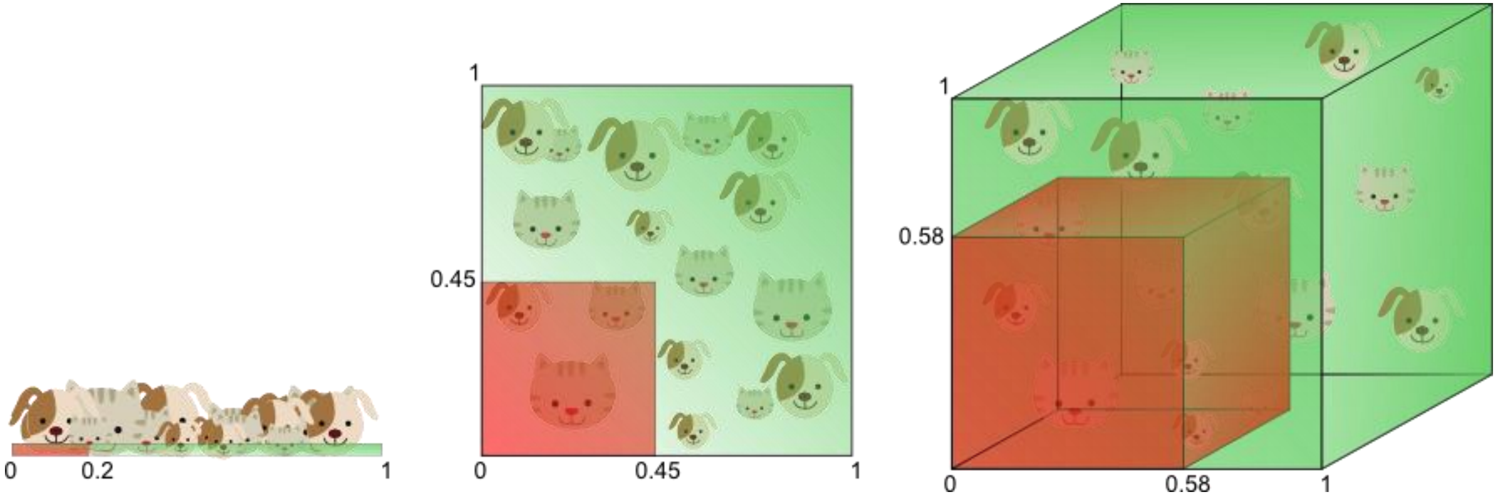




Why Invariance Is A Must For The Rethinking?
Universal Approximation vs. The Curse Of Dimensionality

Universal Approximation vs. The Curse of Dimensionality

- Universal Approximation: “Any 2-layer perceptron can approximate a continuous function to any desired accuracy”.
- The Curse of Dimensionality: The required number of learning samples increases sharply with the dimension, until it is out of feasibility.
- Invariance: Inherent structure of the data, reducing the need for unnecessary and impractical learning, just like from NN to CNN.



- MM Bronstein, J Bruna, T Cohen, et al. Geometric deep learning: Grids, groups, graphs, geodesics, and gauges. *arXiv preprint arXiv:2104.13478,* 2021.

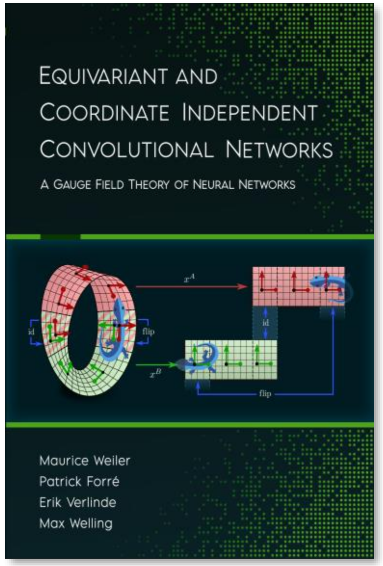
Geometry Deep Learning
Rethinking From The Lens Of Invariance

Geometric Deep Learning

- **Geometric Deep Learning** is a way to rethink deep learning from the lens of invariance:
  - Extending hierarchical invariance to transformations beyond translations. CNN is already invariant to translation, how to generalize this success to rotation, scaling ……
  - Extending hierarchical invariance to data beyond images. CNN works well on images, how to generalize this success to sets, graphs, surfaces ……
  - Harmonizing the existing learning architectures with invariance-principled designs. CNN has translation invariance, how about invariance of LSTM, GNN, Transformer ……

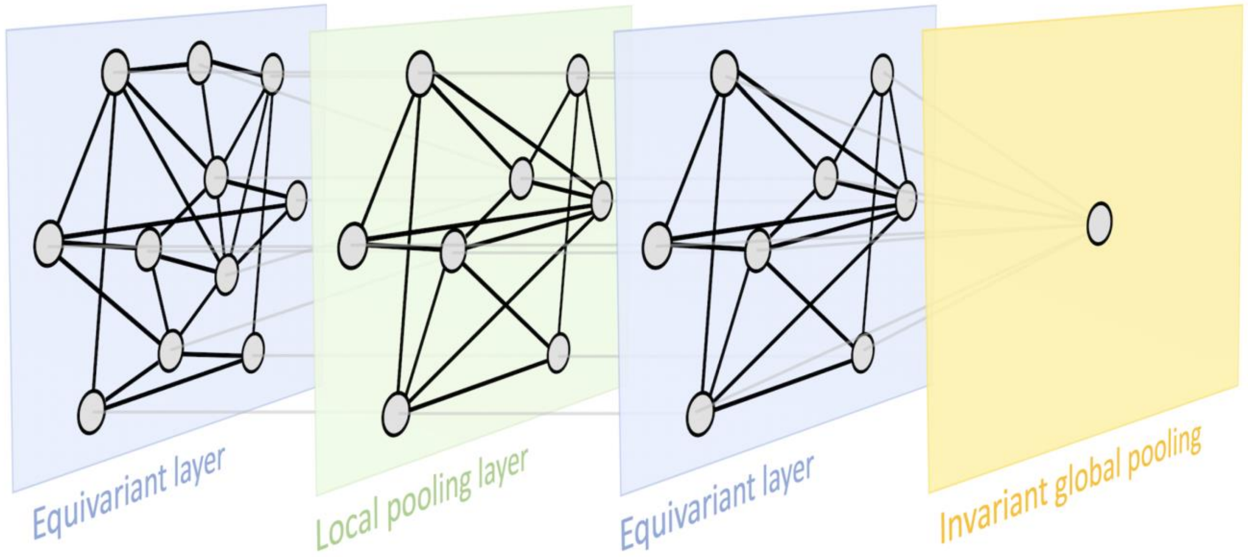
M. M. Bronstein, J. Bruna, T. Cohen, & P. Veličković, 2017

Geometry Deep Learning



Blueprint: High-level Intuition

- A blueprint for achieving a unified hierarchical invariance over different transformations, architectures, and data types.
- Equivariant local representations as the inter-links; invariant global representations as the final-links, which together form a **hierarchical invariant representation**.



- MM Bronstein, J Bruna, T Cohen, et al. Geometric deep learning: Grids, groups, graphs, geodesics, and gauges. *arXiv preprint arXiv:2104.13478,* 2021.

Blueprint: Formalization

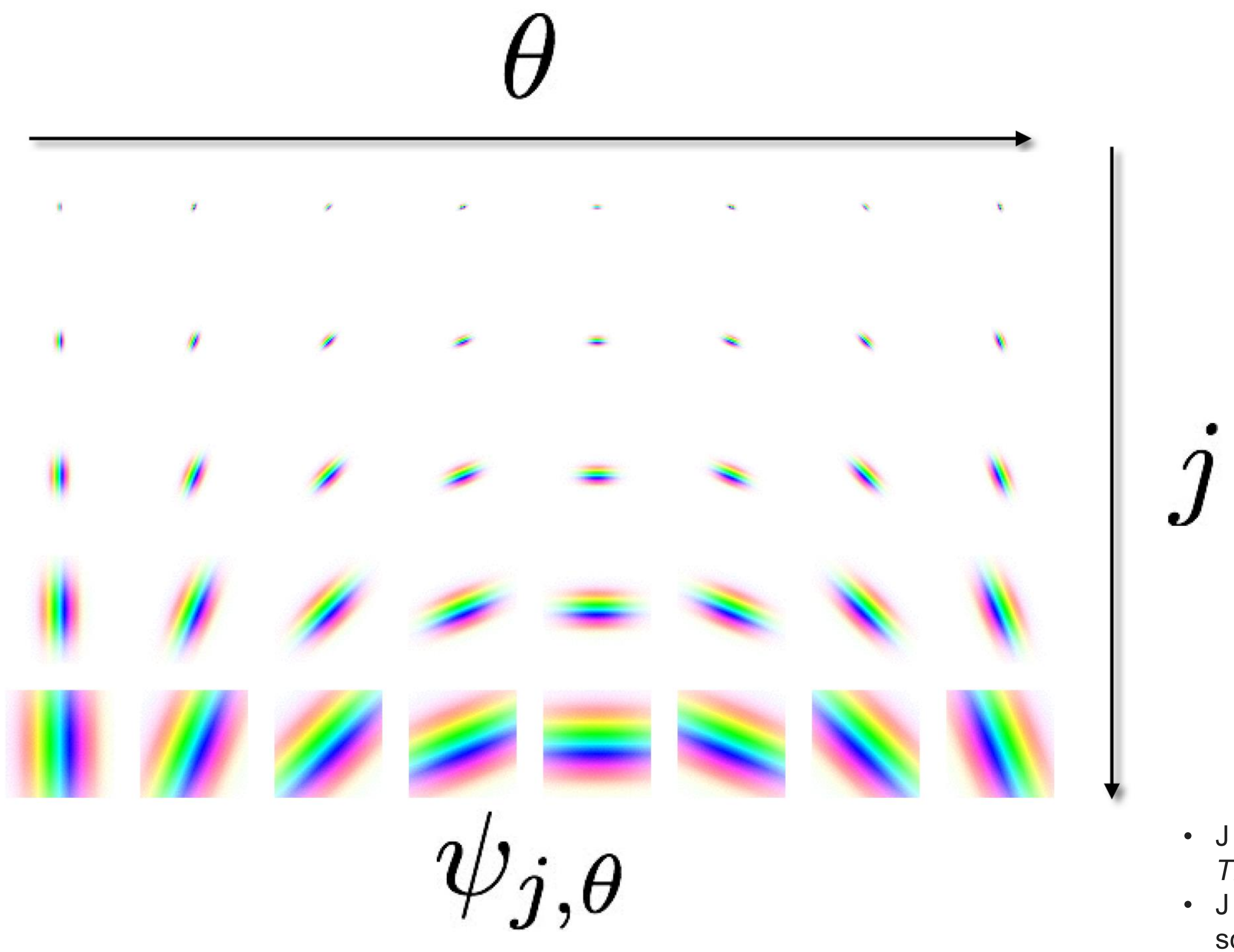
- Let Ω be domain and $\mathfrak{G}$ a symmetry group over Ω.
  - **$\mathfrak{G}$-equivariant layer $\boldsymbol{B}$:** $X\ (\Omega, C) \to X\ (\Omega', C')$, $B(\mathfrak{g} \cdot x) = \mathfrak{g} \cdot B(x)$ for all $\mathfrak{g} \in \mathfrak{G}$ and $x \in X\ (\Omega, C)$.
  - **Nonlinearity $\boldsymbol{\sigma}$:** $C \to C'$ applied element-wise as $(\boldsymbol{\sigma}(x))(u) = \sigma(x(u))$.
  - **Local pooling $\boldsymbol{P}$:** $X\ (\Omega, C) \to X\ (\Omega', C)$, such that $\Omega' \subseteq \Omega$ as a compact version of Ω.
  - **$\mathfrak{G}$-invariant layer $\boldsymbol{A}$:** $X\ (\Omega, C) \to Y$, $A(\mathfrak{g} \cdot x) = A(x)$ for all $\mathfrak{g} \in \mathfrak{G}$ and $x \in X\ (\Omega, C)$.
  - **$\mathfrak{G}$-invariant functions $\boldsymbol{f}$:** $X\ (\Omega, C) \to Y$, $f = A \circ \sigma_N \circ B_N \circ \cdots \circ P_1 \circ \sigma_1 \circ B_1$

Architecture	**Domain Ω**	**Symmetry group $\mathfrak{G}$**
CNN	Grid	Translation
Spherical CNN	Sphere / $\mathrm{SO}(3)$	Rotation $\mathrm{SO}(3)$
Intrinsic / Mesh CNN	Manifold	Isometry $\mathrm{Iso}(\Omega)$ / Gauge symmetry $\mathrm{SO}(2)$
GNN	Graph	Permutation Σ_n
Deep Sets	Set	Permutation Σ_n
Transformer	Complete Graph	Permutation Σ_n
LSTM	1D Grid	Time warping

- MM Bronstein, J Bruna, T Cohen, et al. Geometric deep learning: Grids, groups, graphs, geodesics, and gauges. *arXiv preprint arXiv:2104.13478,* 2021.

Geometric Deep Learning
For Different Transformations

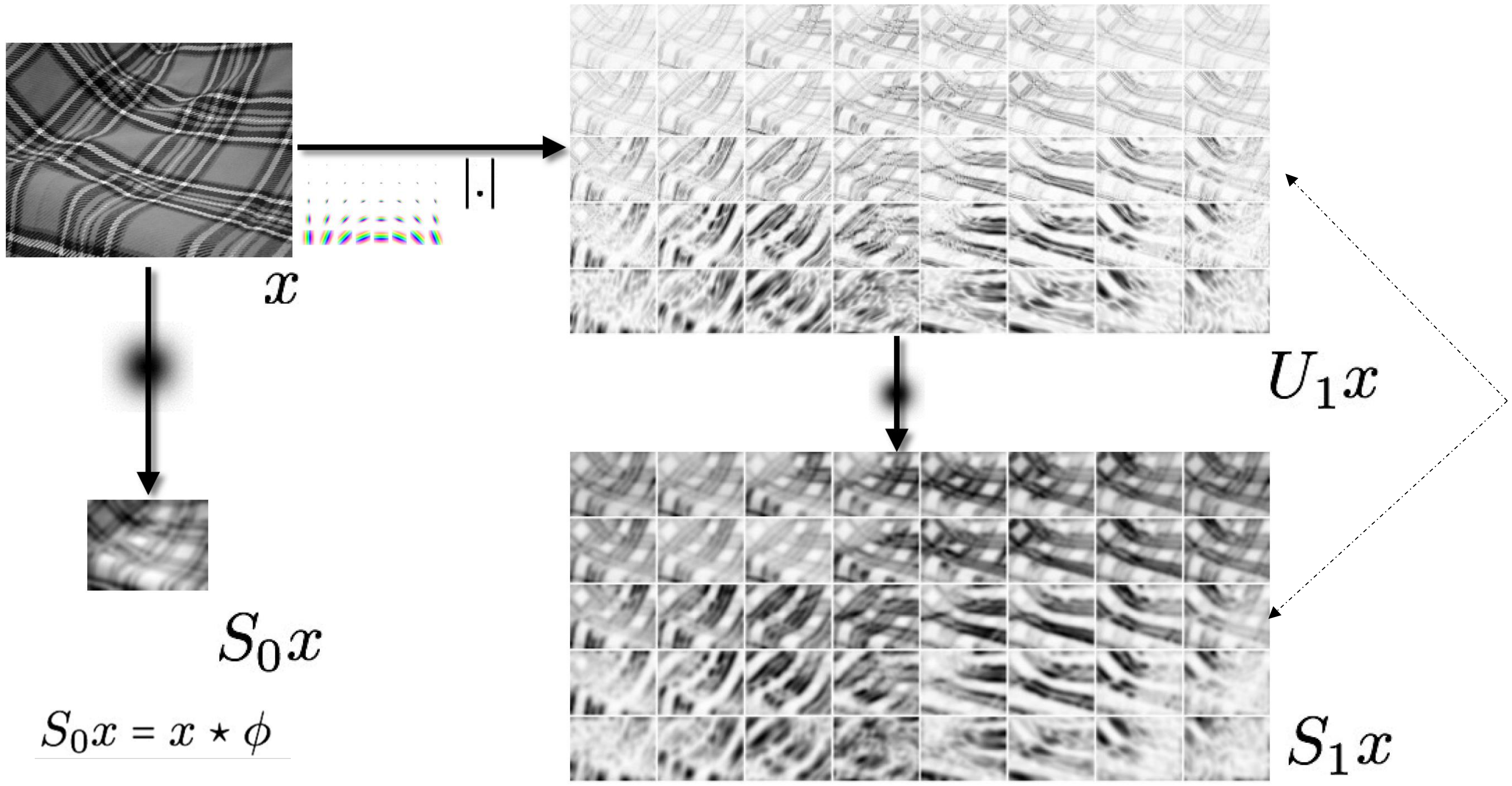
Beyond Translations: Wavelet Scattering Networks



By fully considering the translation, rotation and scaling symmetry group, the wavelet basis functions can be well designed to achieve invariance.

- J Bruna, S Mallat. Invariant scattering convolution networks. *TPAMI,* 2013.
- J Bruna, S Mallat. Rotation, scaling and deformation invariant scattering for texture discrimination. *CVPR,* 2013.

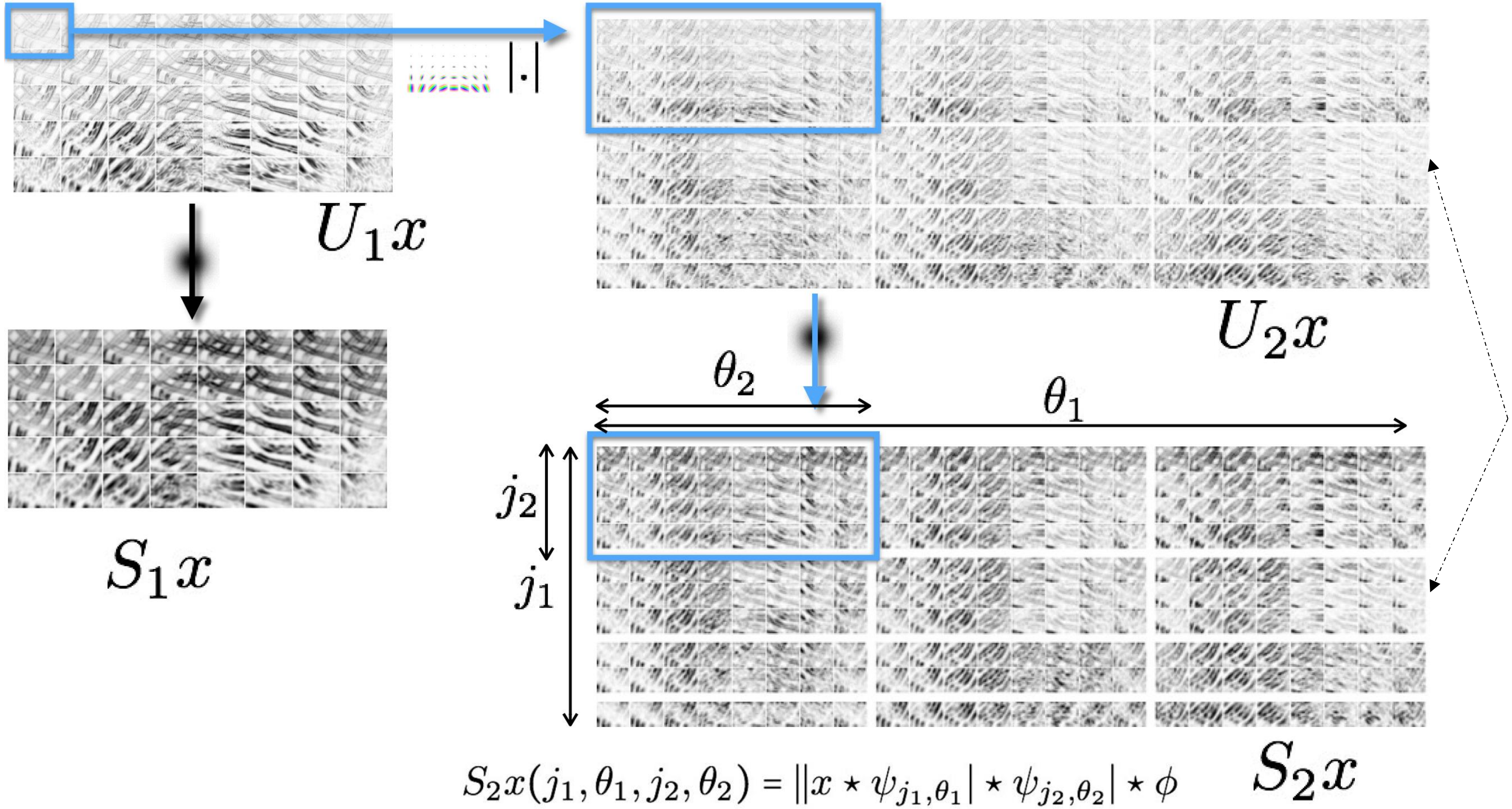
Beyond Translations: Wavelet Scattering Networks



$$S_1 x(j, \theta) = |x \star \psi_{j,\theta}| \star \phi$$

Translation equivariance, rotation and scaling covariance *hold for the wavelet transform.*

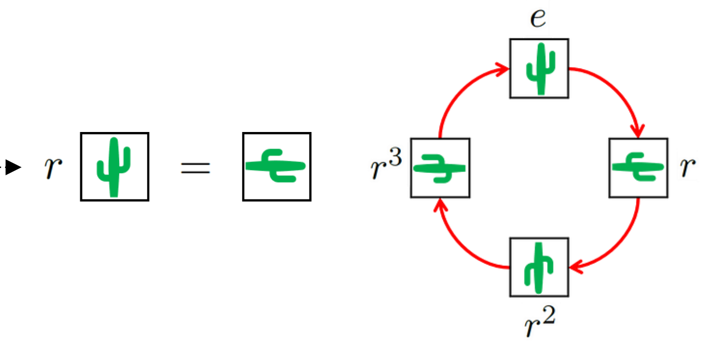
Beyond Translations: Wavelet Scattering Networks



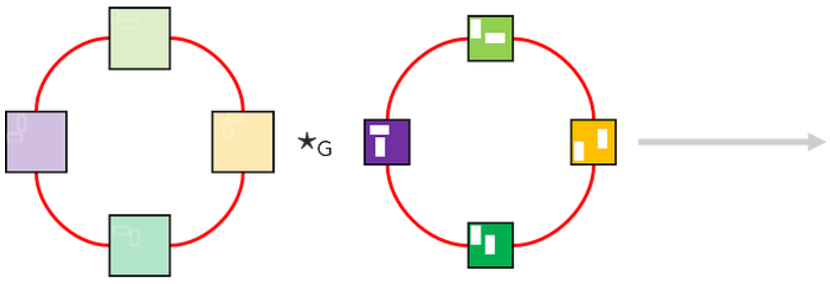
Translation equivariance, rotation and scaling covariance also hold for cascaded wavelet transforms, called Scattering Networks

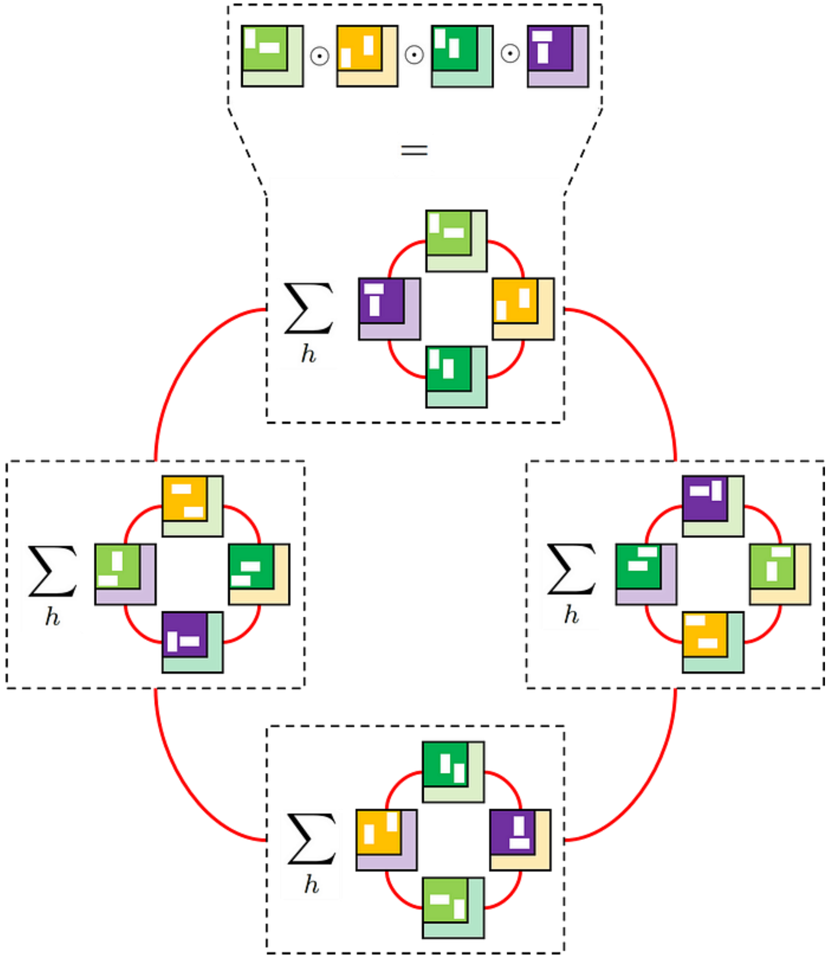
Beyond Translations: Group Equivariant Networks

P4 Group



Convolution On P4 Group





Spatial Rotation + Order Change

$$[f \star \psi](g) = \sum_{h \in G} \sum_{k} f_k(h) \psi_k(g^{-1}h)$$

• T Cohen, M Welling. Group equivariant convolutional networks. *ICML,* 2018.

Beyond Translations: Group Equivariant Networks

$$[f \star \psi^i](x) = \sum_{\boxed{y \in \mathbb{Z}^2}} \sum_{k=1}^{K^l} f_k(y)\psi_k^i(\boxed{y - x})$$

Convolution

$$\begin{aligned}
[[L_t f] \star \psi](x) &= \sum_y f(y - t)\psi(y - x) \\
&= \sum_y f(y)\psi(y + t - x) \\
&= \sum_y f(y)\psi(y - (x - t)) \\
&= [L_t[f \star \psi]](x).
\end{aligned}$$

Translation Equivariance

$$[f \star \psi](g) = \sum_{\boxed{h \in G}} \sum_k f_k(h)\psi_k(\boxed{g^{-1}h})$$

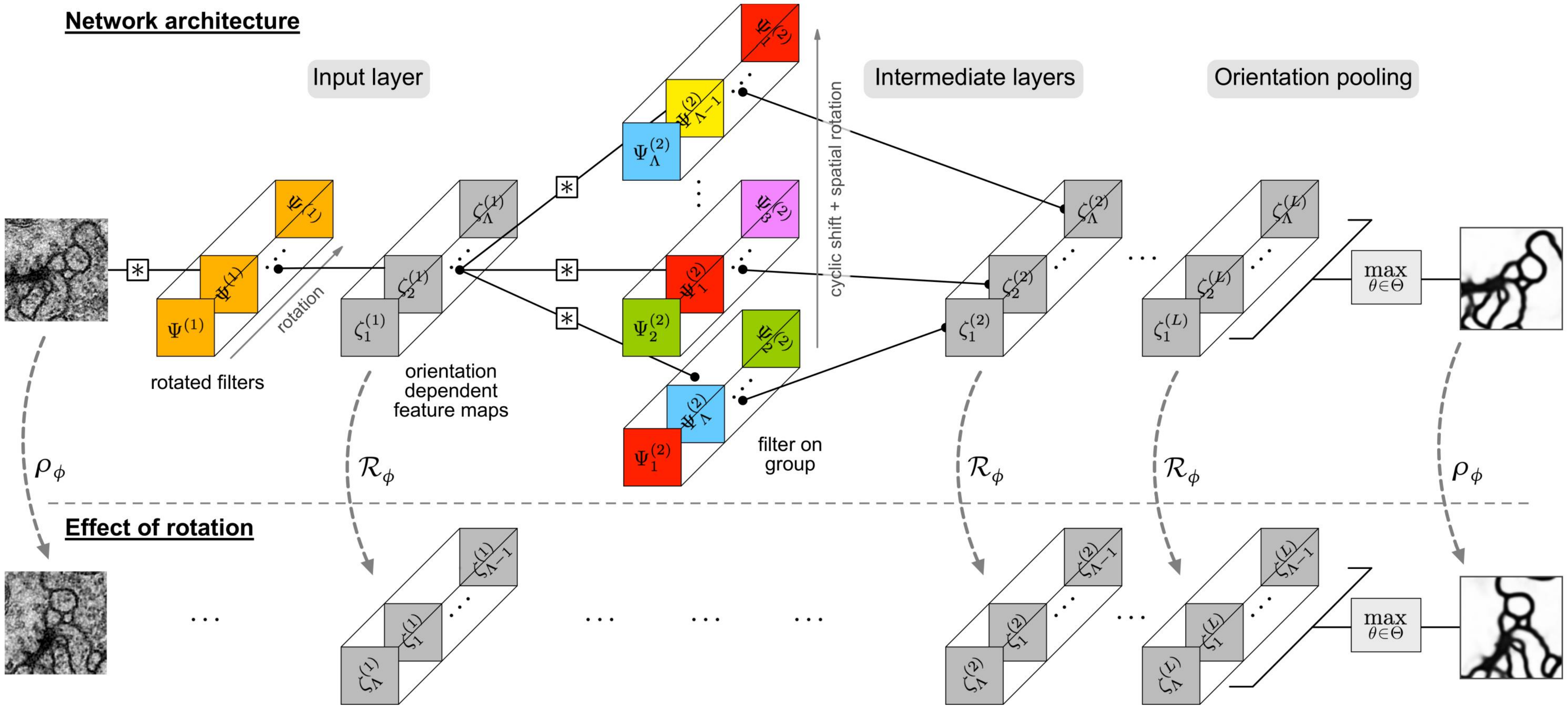
Group Convolution

$$\begin{aligned}
[[L_u f] \star \psi](g) &= \sum_{h \in G} \sum_k f_k(u^{-1}h)\psi(g^{-1}h) \\
&= \sum_{h \in G} \sum_k f(h)\psi(g^{-1}uh) \\
&= \sum_{h \in G} \sum_k f(h)\psi((u^{-1}g)^{-1}h) \\
&= [L_u[f \star \psi]](g)
\end{aligned}$$

Group Equivariance

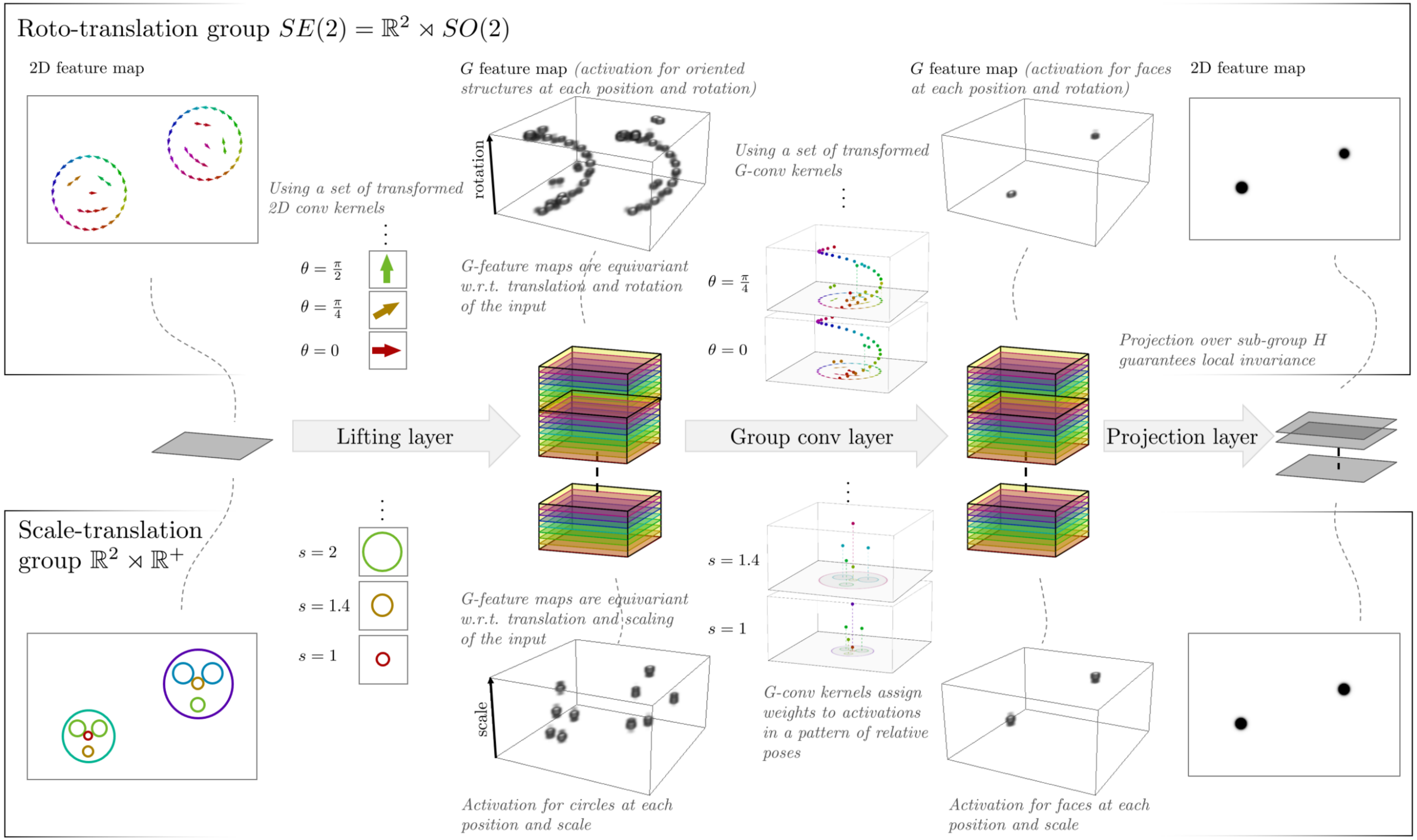
- T Cohen, M Welling. Group equivariant convolutional networks. *ICML,* 2018.

Beyond Translations: Group Equivariant Networks



- M Weiler, FA Hamprecht, M Storath. Learning steerable filters for rotation equivariant CNNs. *CVPR,* 2018.

Beyond Translations: Group Equivariant Networks



- EJ Bekkers. B-spline CNNs on lie groups. *ICLR,* 2020.

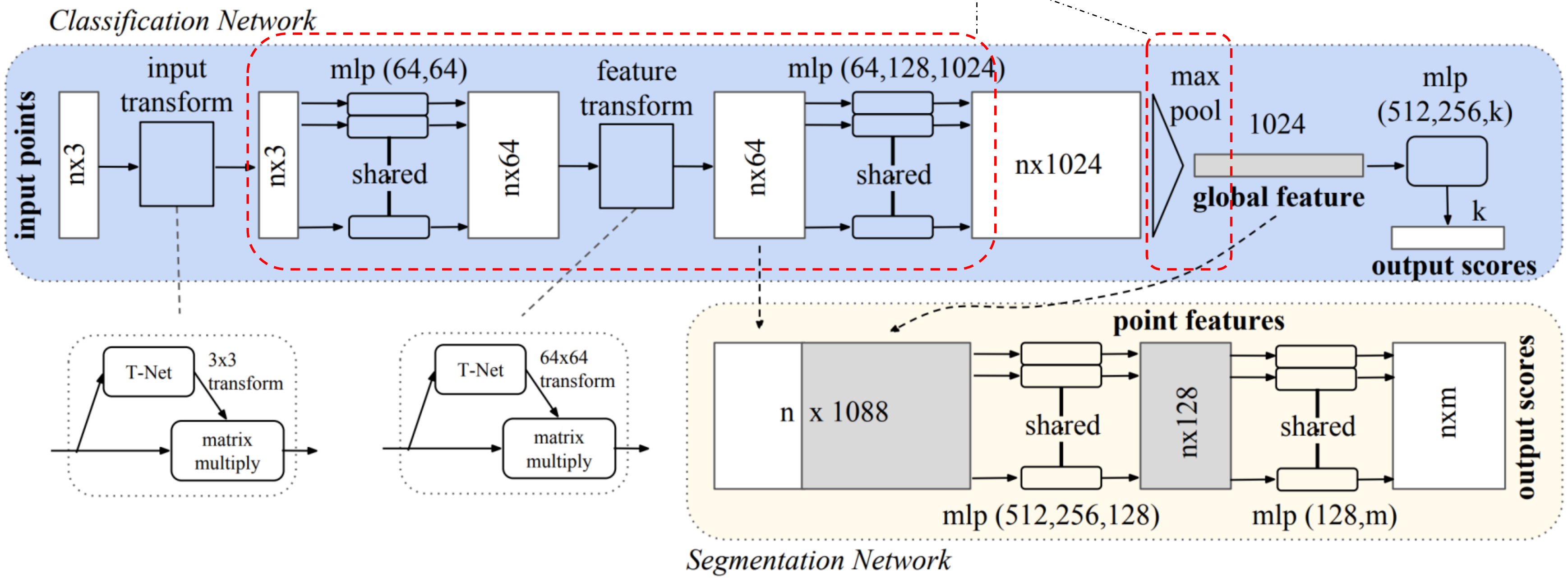
Geometric Deep Learning
For Different Architectures And Data Types

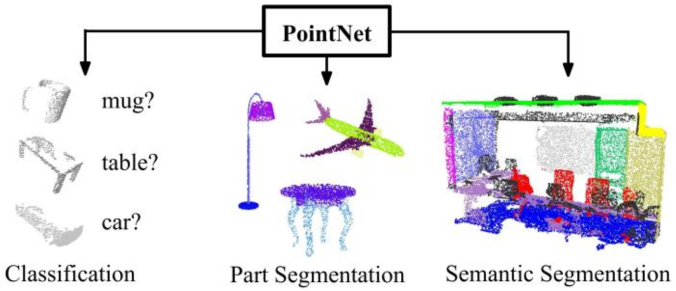
Beyond Images: Deep Sets and PointNet

$$f(\{x_1, \ldots, x_M\}) = f(\{x_{\pi(1)}, \ldots, x_{\pi(M)}\})$$
Permutation Invariance on Set

$$\mathbf{f}([x_{\pi(1)}, \ldots, x_{\pi(M)}]) = [f_{\pi(1)}(\mathbf{x}), \ldots, f_{\pi(M)}(\mathbf{x})]$$
Permutation Equivariance on Set

$$\rho\left(\sum_{x \in X} \phi(x)\right)$$
Deep Sets, Point-wise ϕ for Permutation Equivariance and Global Pooling Σ for Permutation Invariance

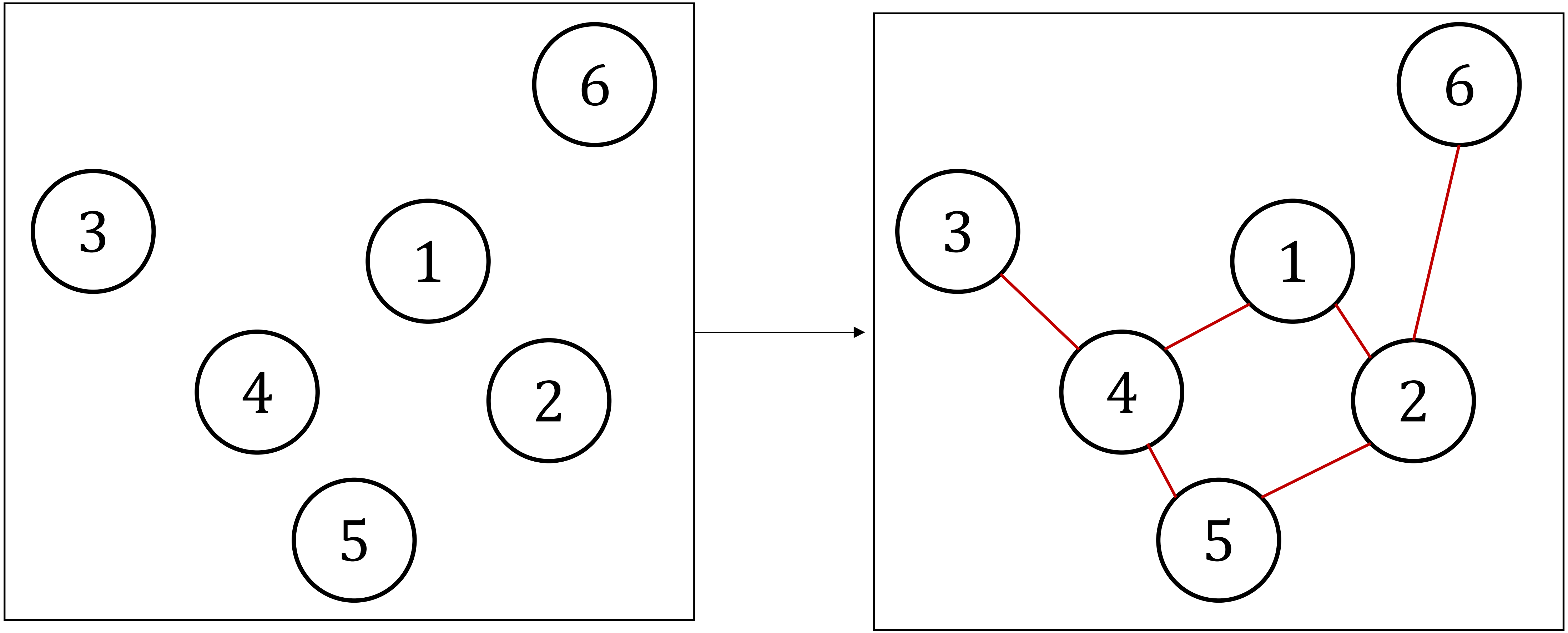




PointNet: A Practice of Deep Sets

- M Zaheer, S Kottur, S Ravanbakhsh. Deep sets. *NIPS,* 2017.
- CR Qi, H Su, K Mo, et al. PointNet: Deep learning on point sets for 3D classification and segmentation. *CVPR,* 2017.

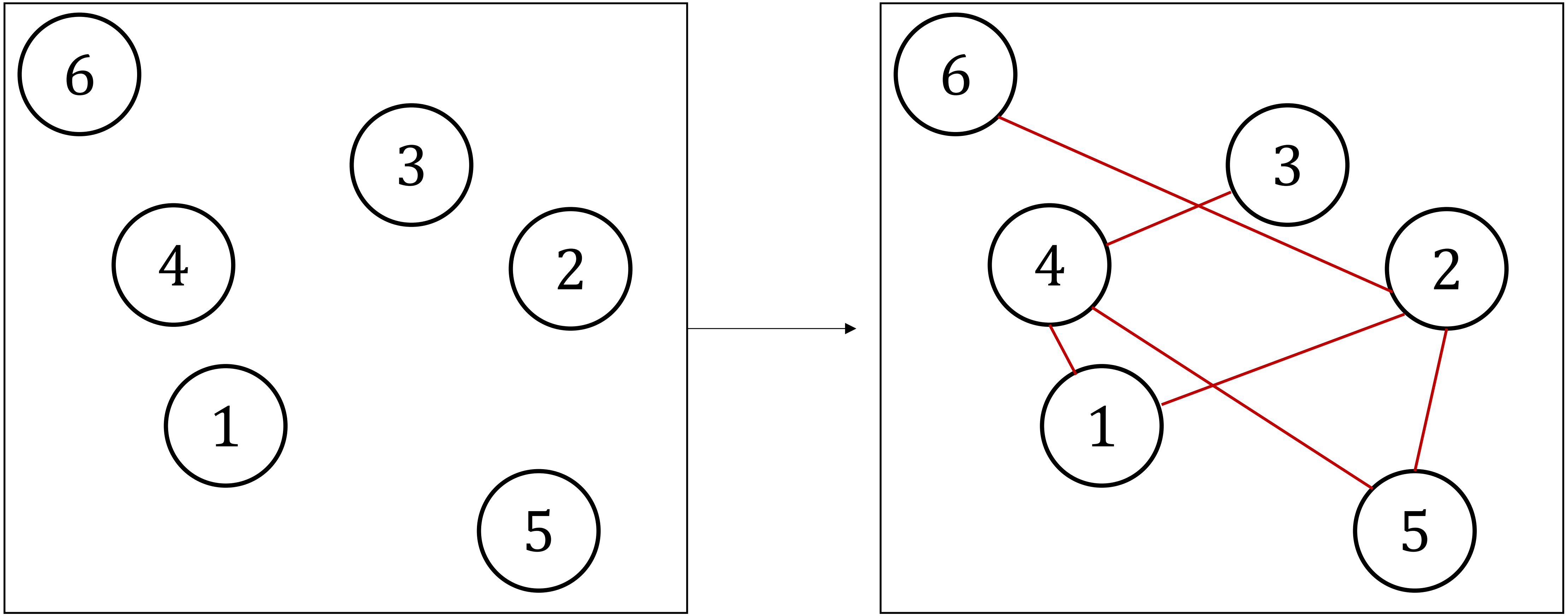
Beyond Images: Graph Networks



Set: $S = \{1,2, \ldots, 6\}$

Graph: $G = \{X, A\}$,
$X = S, A = \{\{1,2\}, \ldots \{2,6\}\}$

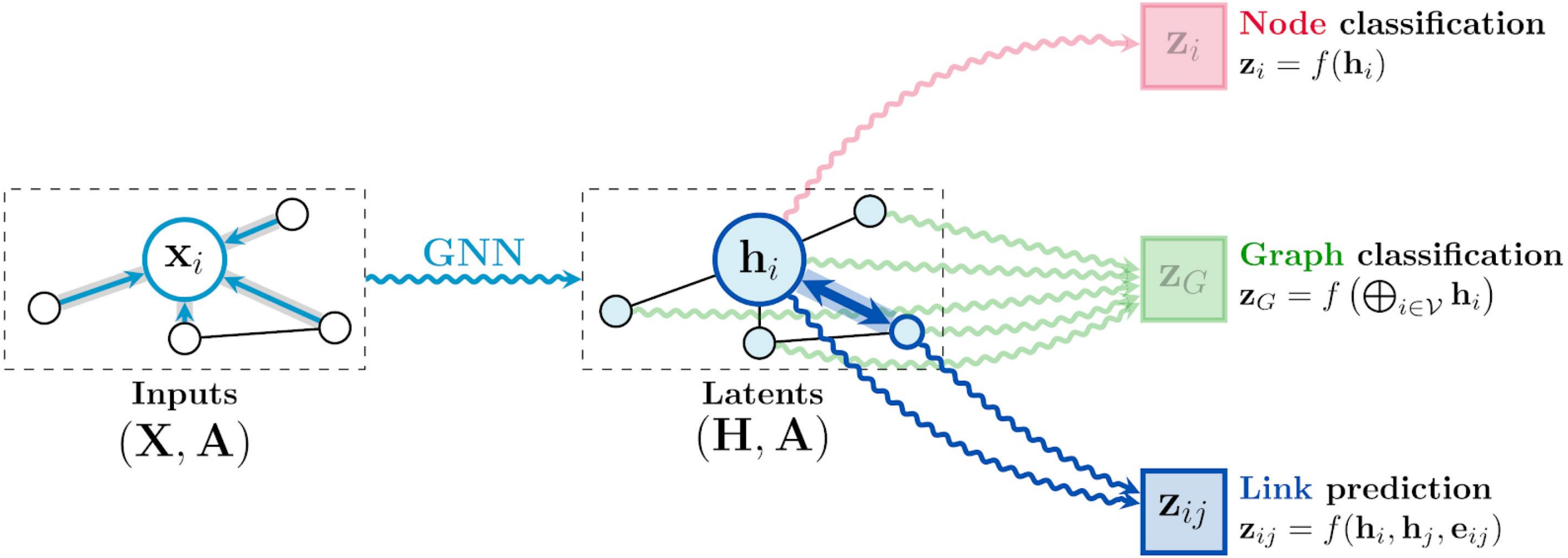
Beyond Images: Graph Networks



Set Permutation
Just order changes

Graph Permutation
Node order changes with edge order changes

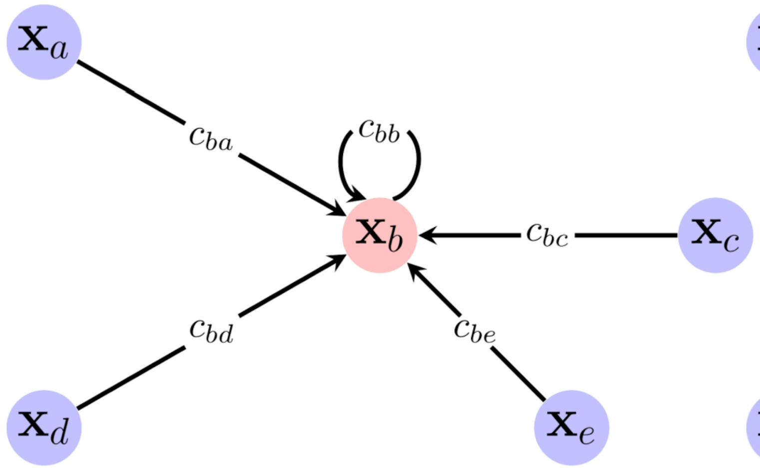
Beyond Images: Graph Networks



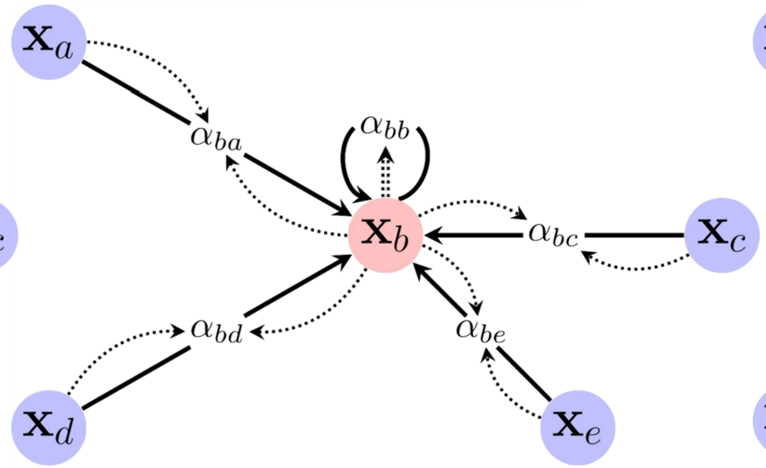
Local and Global Applications of GNN

- TN Kipf, M Welling. Semi-supervised classification with graph convolutional networks. *ICLR,* 2017.
- P Veličković, G Cucurull, A Casanova, et al. Graph attention networks. *ICLR,* 2018.
- J Gilmer, SS Schoenholz, PF Riley, et al. Neural message passing for quantum chemistry. *ICML,* 2017.

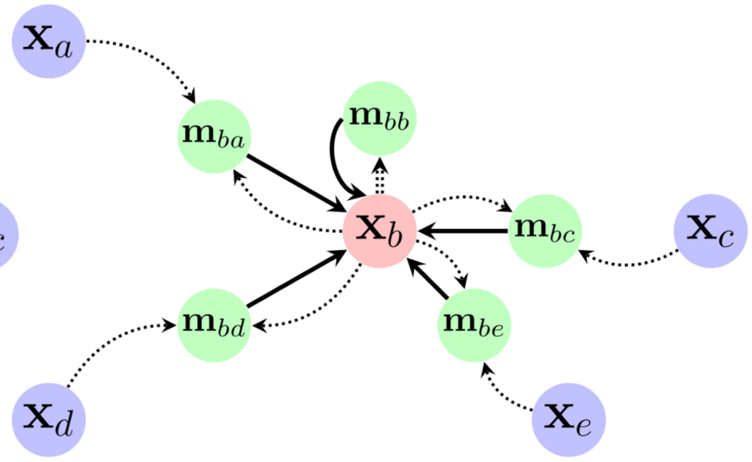
Beyond Images: Graph Networks



Convolutional



Attentional



Message-passing

$$\mathbf{h}_i = \phi\left(\mathbf{x}_i, \bigoplus_{j\in\mathcal{N}_i} c_{ij}\psi(\mathbf{x}_j)\right)$$

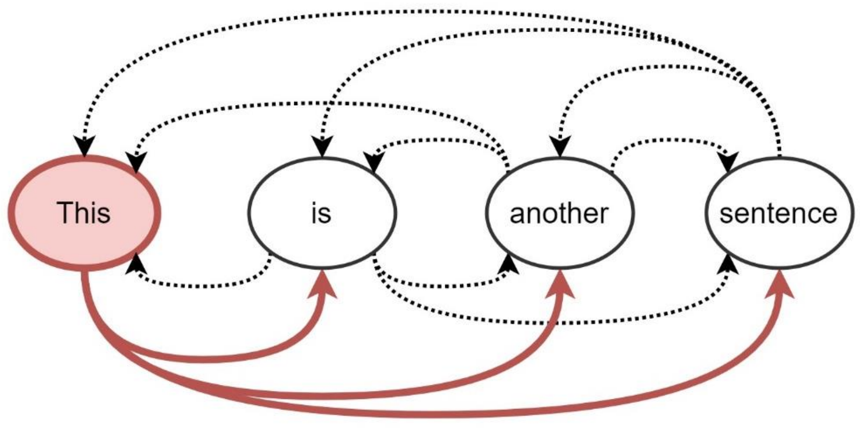
$$\mathbf{h}_i = \phi\left(\mathbf{x}_i, \bigoplus_{j\in\mathcal{N}_i} a(\mathbf{x}_i,\mathbf{x}_j)\psi(\mathbf{x}_j)\right)$$

$$\mathbf{h}_i = \phi\left(\mathbf{x}_i, \bigoplus_{j\in\mathcal{N}_i} \psi(\mathbf{x}_i,\mathbf{x}_j)\right)$$

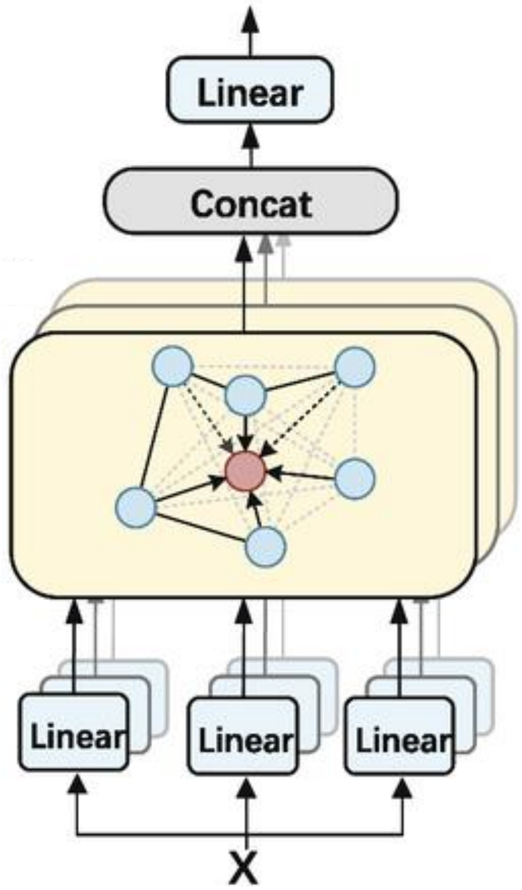
Permutation Equivariant GNN Layers

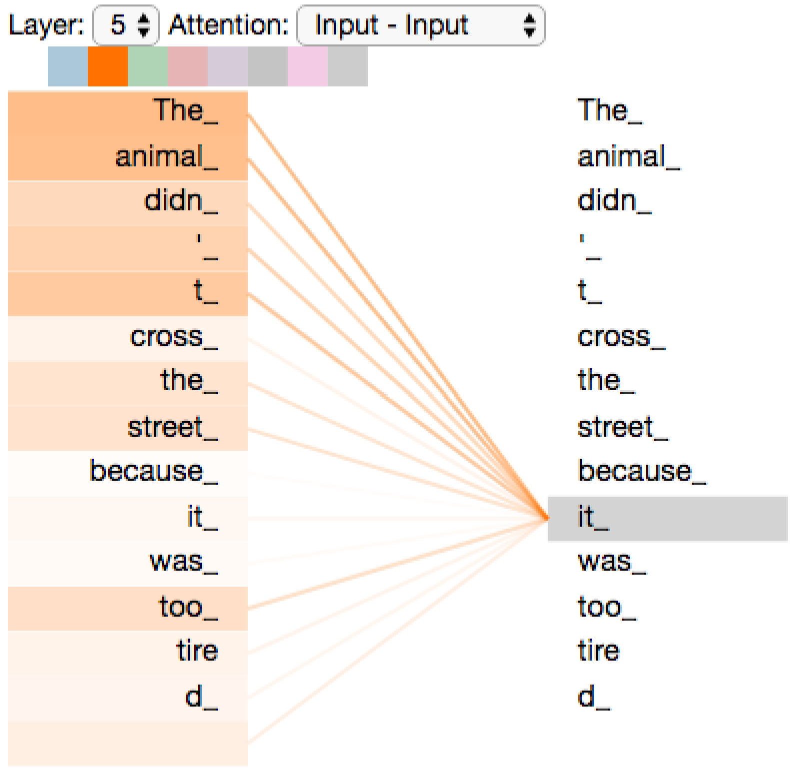
- TN Kipf, M Welling. Semi-supervised classification with graph convolutional networks. *ICLR,* 2017.
- P Veličković, G Cucurull, A Casanova, et al. Graph attention networks. *ICLR,* 2018.
- J Gilmer, SS Schoenholz, PF Riley, et al. Neural message passing for quantum chemistry. *ICML,* 2017.

Beyond Images: Transformers



$$\mathbf{h}_u = \phi\left(\mathbf{x}_u, \bigoplus_{v \in \mathcal{V}} a(\mathbf{x}_u, \mathbf{x}_v)\psi(\mathbf{x}_v)\right)$$





Transformers are GNNs on Fully-connected Graph

- CK Joshi. https://thegradient.pub/transformers-are-graph-neural-networks/

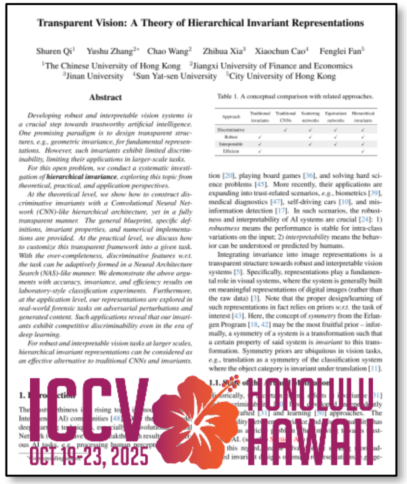
Our Contributions

	Moment Invariants	Multiscale and Wavelet	CNN to Geometry Deep Learning
	2000s	*2010s*	*2020s*
	Flusser	Lowe Lindeberg Mallat…	LeCun Bronstein…
Assumption	Global	Local	Hierarchical
Invari./Discri.	Invariance	Mid?	Discriminability
Robust./Interp./Effic.	Good	Mid?	Not Good
Our Contributions	CSUR’23 TPAMI’24 TIFS review	TPAMI’23 USENIX Security’25 TIP’25	ICCV’25 TIP’25

From Global And Local To Hierarchical

Exploring Hierarchical Invariants

- Reviewing the above hierarchical invariants, one can note a gap: equivariant CNNs are discriminative and invariant, but are implemented by sampling of symmetry, with limited efficiency and invariance, especially for joint invariance.
- We tried to define hierarchical (discriminative) invariants while being one-shot. We achieved this goal by exploring the potential of **classical moment invariants**.



Transparent Vision: A Theory of Hierarchical Invariant Representations

Shuren Qi[1] Yushu Zhang[2*] Chao Wang[2] Zhihua Xia[3] Xiaochun Cao[4] Fenglei Fan[5]

[1]The Chinese University of Hong Kong [2]Jiangxi University of Finance and Economics [3]Jinan University [4]Sun Yat-sen University [5]City University of Hong Kong

Abstract

Developing robust and interpretable vision systems is a crucial step towards trustworthy artificial intelligence. One promising paradigm is to design transparent structures, e.g., geometric invariance, for fundamental representations. However, such invariants exhibit limited discriminability, limiting their applications in larger-scale tasks.

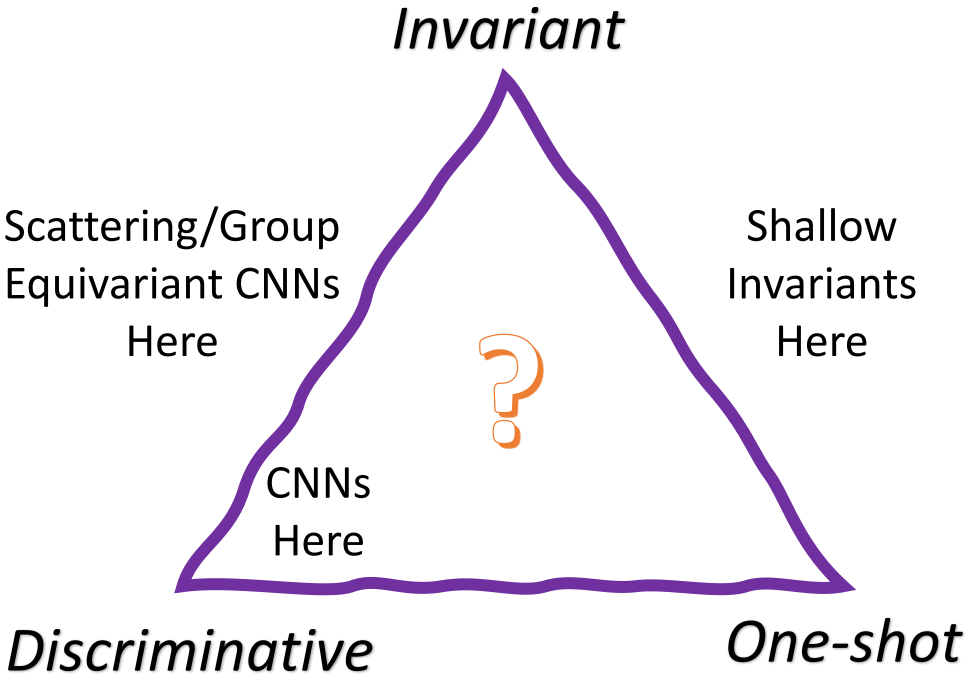
*For this open problem, we conduct a systematic investigation of **hierarchical invariance**, exploring this topic from*

Table 1. A conceptual comparison with related approaches.

Approach	Traditional invariants	Traditional CNNs	Scattering networks	Equivariant networks	Hierarchical invariants
Discriminative		✓	✓	✓	✓
Robust	✓		✓	✓	✓
Interpretable	✓		✓	✓	✓
Efficient	✓				✓

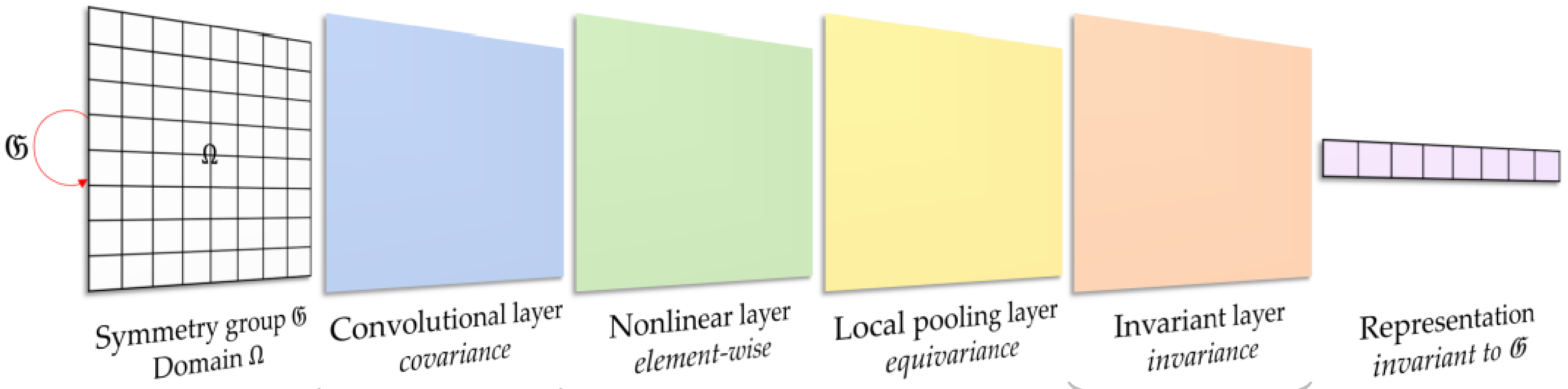
tion [20], playing board games [36], and solving hard sci-

- S. Qi, Y. Zhang, C. Wang, et al. Transparent Vision: A Theory of Hierarchical Invariant Representations. *ICCV*, 2025.



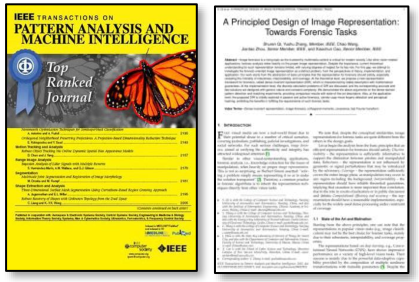
Blueprint

- First, we rethink the typical modules of CNN, unifying the theory of global and local invariants into a hierarchical network.

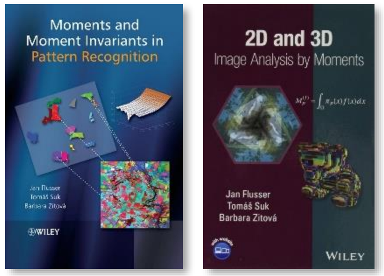


Need Local Invariants

Need Global Invariants



Recalling the geometric deep learning blueprint, we are surprised that we already have the components to form the hierarchical invariance, we just have not yet assembled them.

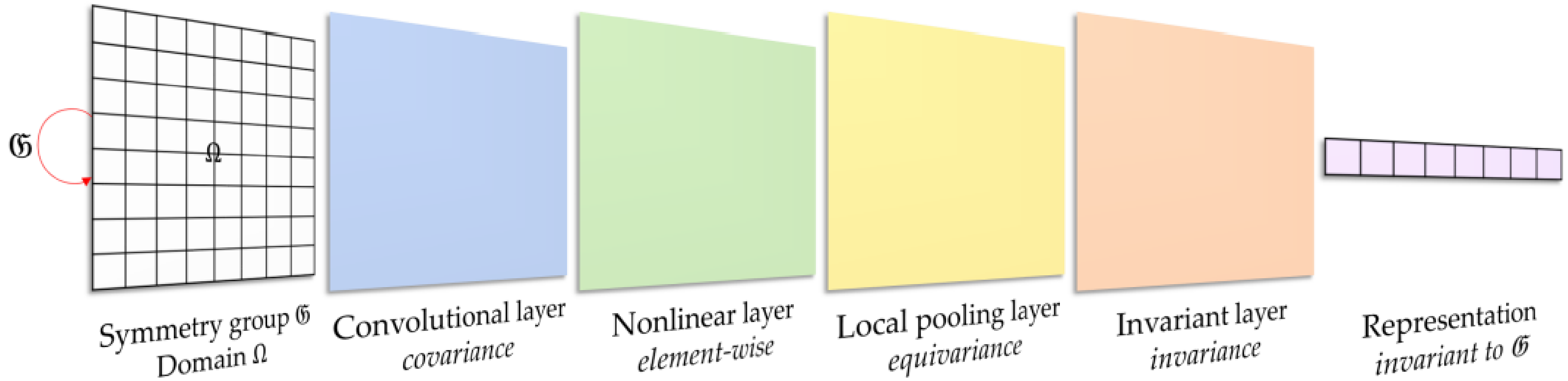


- S. Qi, Y. Zhang, C. Wang, et al. A Principled Design of Image Representation: Towards Forensic Tasks. *TPAMI*, 2023.

- J. Flusser, B. Zitova, T. Suk. *Moments and Moment Invariants in Pattern Recognition*. John Wiley & Sons, 2009.

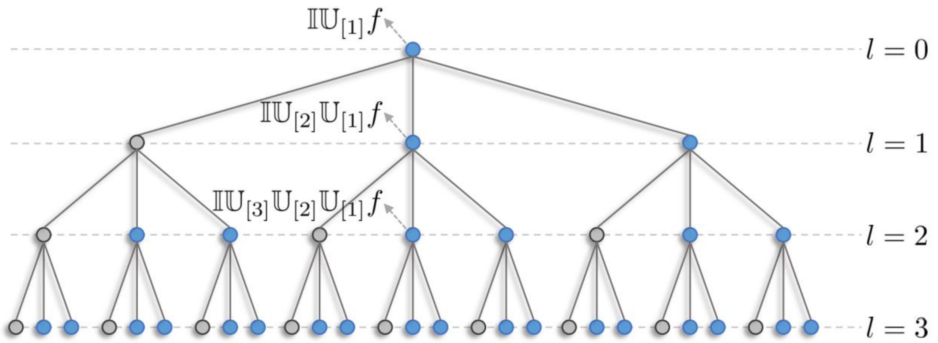
Definition

- Then, we can define new modules with their cascades to fulfill the blueprint:
  - Ω is 2D grid for images; $\mathfrak{G}$ is a translation, rotation, flipping, and scaling symmetry group over Ω.
  - **$\mathfrak{G}$-covariant convolutional layer:** $\mathbb{C}M \triangleq \langle M, V_{nm}^{uvw}\rangle = M(i,j;k) \otimes (H_{nm}^{w}(i,j))^T$
  - **Nonlinearity layer:** $\mathbb{S}M = \sigma(M(i,j)) \triangleq |M(i,j;k)|$
  - **Local pooling layer:** $\mathbb{P}M = M'$
  - **$\mathfrak{G}$-invariant layer:** $\mathbb{I}M = \mathcal{I}(\{\langle M(i,j;k), V_{nm}(x_i,y_j)\rangle\})$
  - **$\mathfrak{G}$-invariant representation:** $\mathcal{R}_p \triangleq \mathbb{I} \circ \mathbb{P}_{[L]} \circ \mathbb{S}_{[L]} \circ \mathbb{C}_{[L]} \circ \cdots \circ \mathbb{P}_{[1]} \circ \mathbb{S}_{[1]} \circ \mathbb{C}_{[1]}$

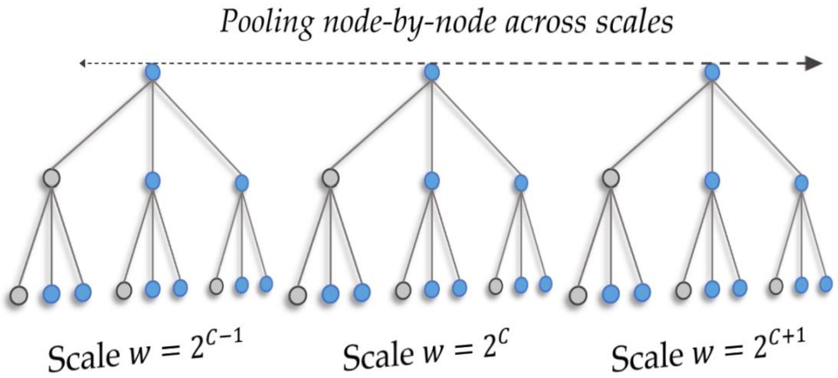


Property

- The group theory shows the one-shot symmetry property at each inter layer:
  - $\mathfrak{G}_1$ is the translation, rotation, and flipping symmetry group; $\mathfrak{G}_2$ is a scaling symmetry group, with scaling factor s. Any $\mathfrak{G}_0 \subseteq \mathfrak{G}_1 \times \mathfrak{G}_2$ as the symmetry group of interest. A representation unit denoted as $\mathbb{U} \triangleq \mathbb{P} \circ \mathbb{S} \circ \mathbb{C}$.
  - $\mathfrak{G}_1$ **Equivariance:** $\mathbb{U}_{[L]} \circ \cdots \circ \mathbb{U}_{[2]} \circ \mathbb{U}_{[1]}(\mathfrak{g}_1 M) \equiv \mathfrak{g}_1 \mathbb{U}_{[L]} \circ \cdots \circ \mathbb{U}_{[2]} \circ \mathbb{U}_{[1]}(M)$
  - $\mathfrak{G}_2$ **Covariance:** $\mathbb{U}^w_{[L]} \circ \cdots \circ \mathbb{U}^w_{[2]} \circ \mathbb{U}^w_{[1]}(\mathfrak{g}_2 M) \equiv \mathfrak{g}'_2 \mathbb{U}^w_{[L]} \circ \cdots \circ \mathbb{U}^w_{[2]} \circ \mathbb{U}^w_{[1]}(M) \quad \mathfrak{g}'_2 \mathbb{U}^w \triangleq \mathfrak{g}_2 \mathbb{U}^{ws}$
  - $\mathfrak{G}_0$ **Hierarchical Invariance:** $\mathbb{I}(\mathfrak{g}'_0 M)_{[L]} \equiv \mathbb{I} M_{[L]}$



A Single-scale Practice with $\mathfrak{G}_1$ Invariance

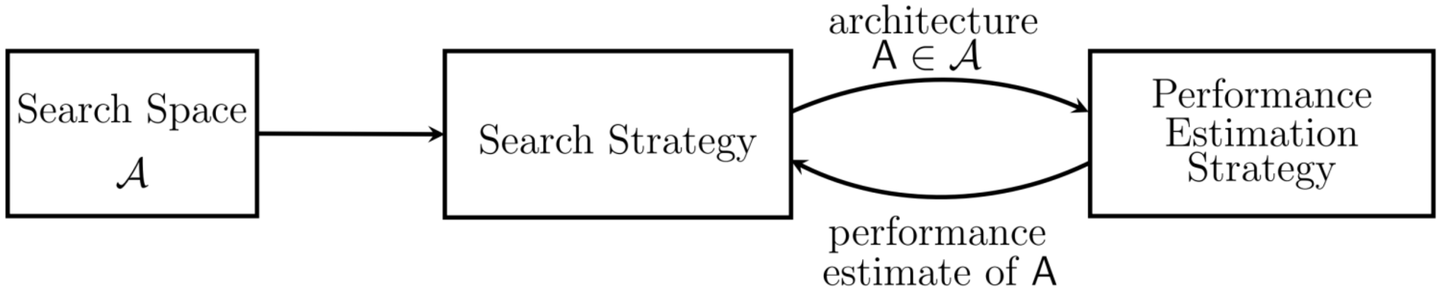


A Multi-scale Practice with $\mathfrak{G}_0$ Invariance

Practice

How to select task-discriminative features from such a huge feature space?

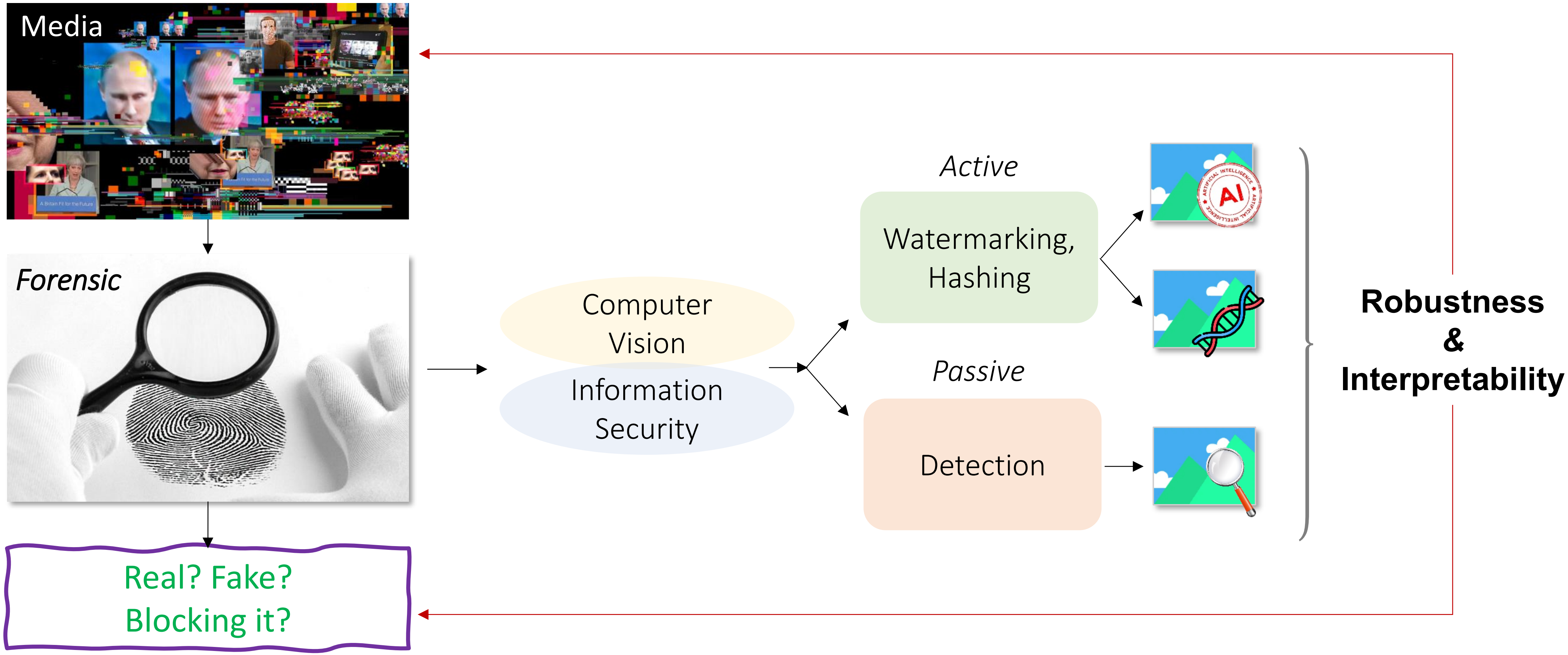
- Feature/Architecture Selection, inspired by Neural Architecture Search (NAS)
  - **Super network** covers preferred and sufficiently diverse parameters.
  - **Correlation analysis** for filtering the most relevant features.
  - **Concise network** by resampling the super network for most relevant features.
- Cascading Learning Module, inspired by Hybrid Representation Learning (HRL)
  - Replacing shallow layers of learning CNNs with our layers, such that discriminative features are formed in a space with geometric symmetries.



Our Contributions

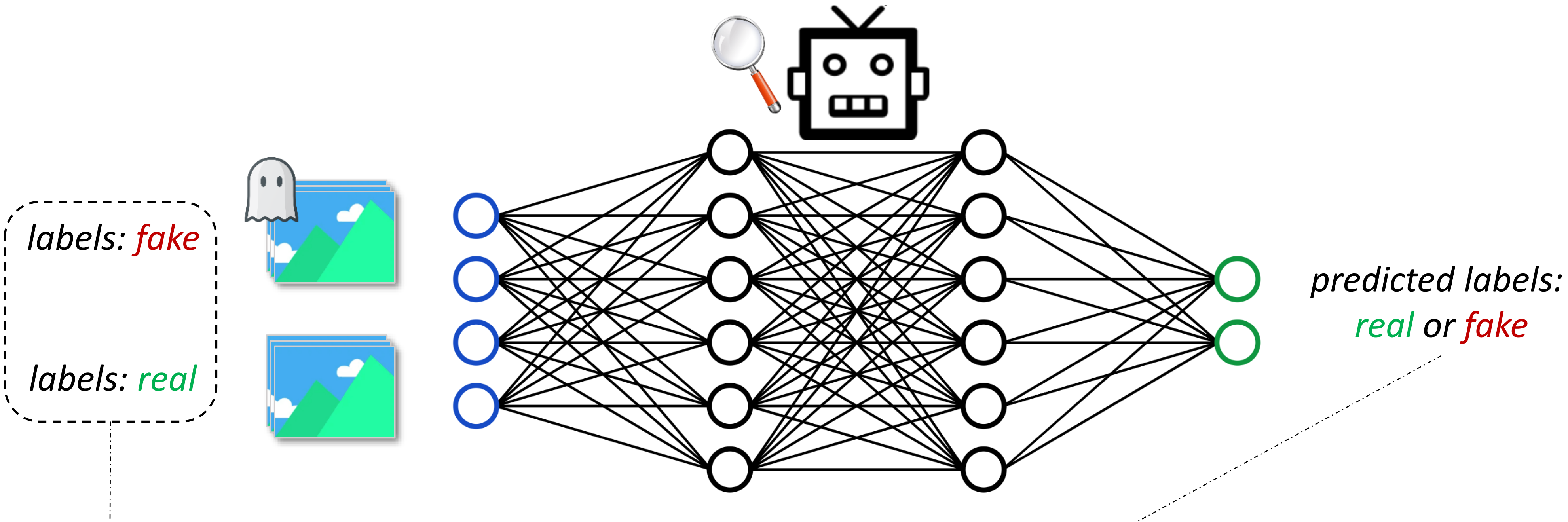
	Moment Invariants	Multiscale and Wavelet	CNN to Geometry Deep Learning
	2000s	*2010s*	*2020s*
	Flusser	Lowe Lindeberg Mallat…	LeCun Bronstein…
Assumption	Global	Local	Hierarchical
Invari./Discri.	Invariance	Mid?	Discriminability
Robust./Interp./Effic.	Good	Mid?	Not Good
Our Contributions	CSUR'23 TPAMI'24 TIFS review	TPAMI'23 USENIX Security'25 TIP'25	ICCV'25 TIP'25

Forensic, Fighting Against AIGC Abuse



- T Wang, Y Zhang, S Qi, et al. Security and privacy on generative data in AIGC: A survey. *ACM Computing Surveys*, 2024.

AIGC Detection: Motivations

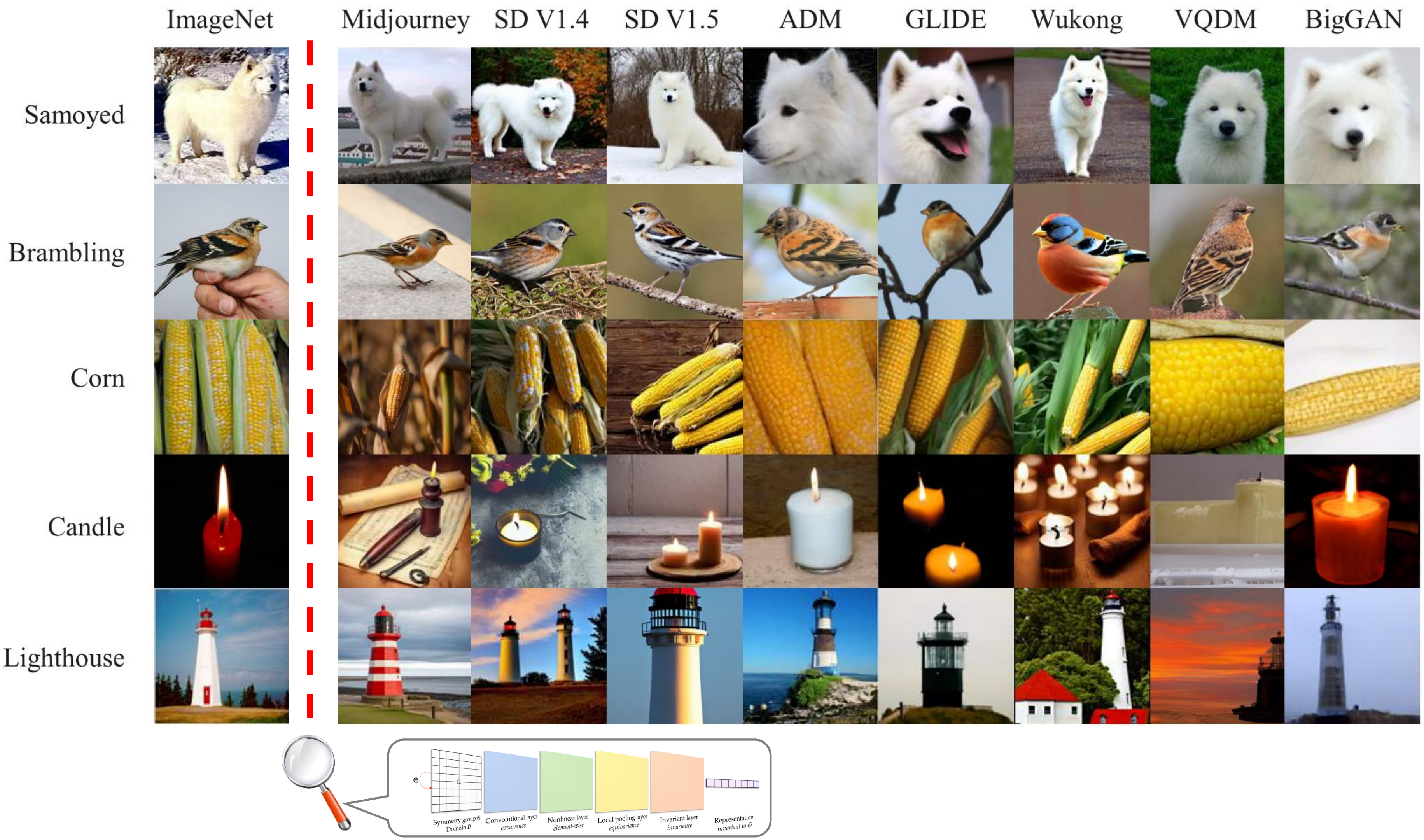


$$\mathbb{E}_{\mathbf{x},\mathbf{y}\in S}[\ell(\mathbf{y}, h(\mathbf{x}))]$$

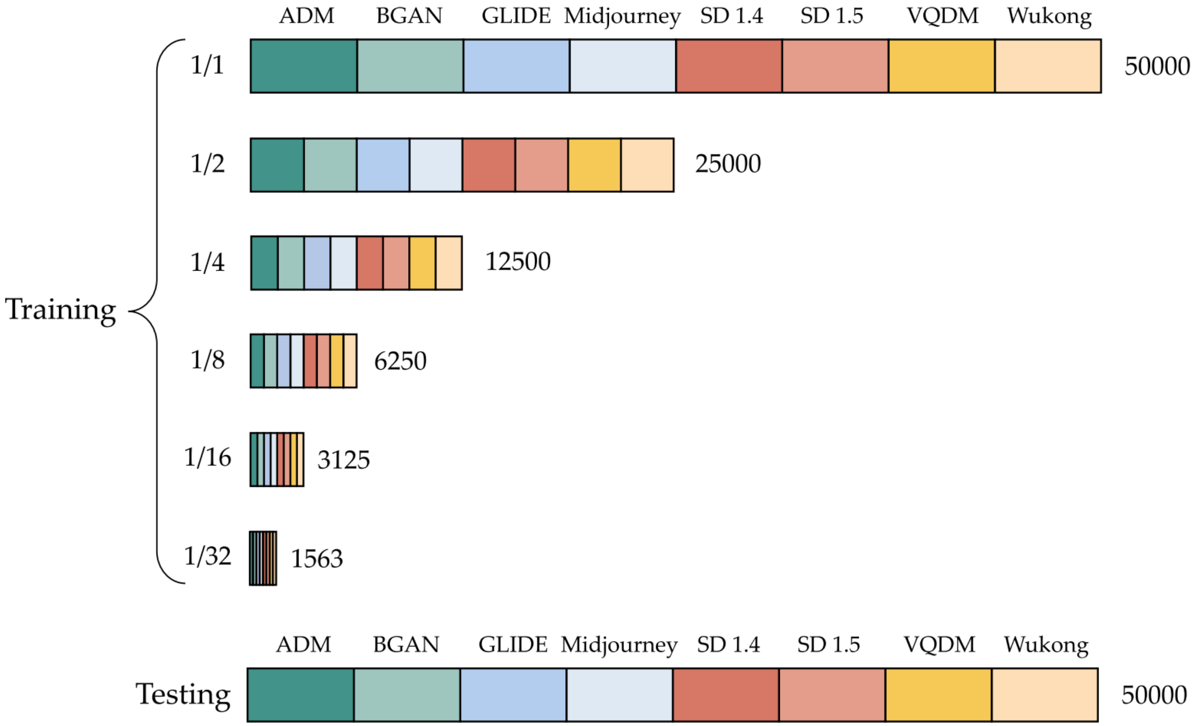
Where is the Robustness and Interpretability?

- S. Qi, C. Wang, Z. Huang, et al. Boosting Geometric Invariants for Discriminative Forensics of Large-Scale Generated Visual Content. *IEEE TIP* 2025.

AIGC Detection: Ideas



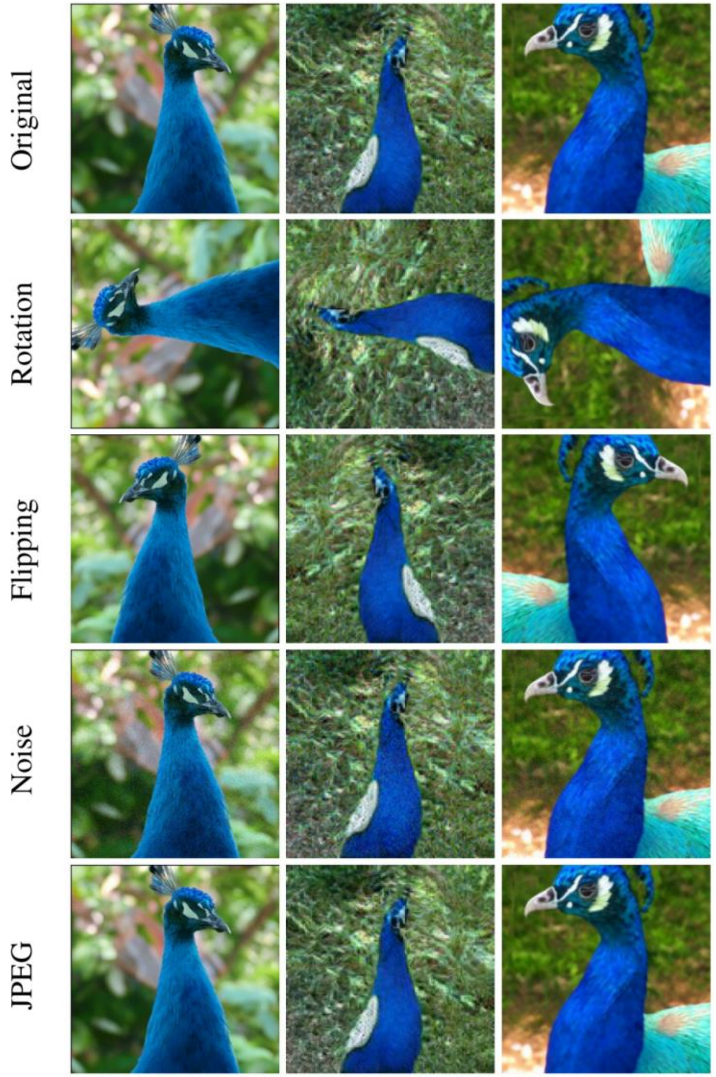
AIGC Detection: Training and Sample Efficiency



	1/1	1/2	1/4	1/8	1/16	1/32
BGI NN	95.30	96.01	95.14	95.39	95.73	94.23
BGI SVM	96.23	96.06	96.49	96.09	95.71	95.08

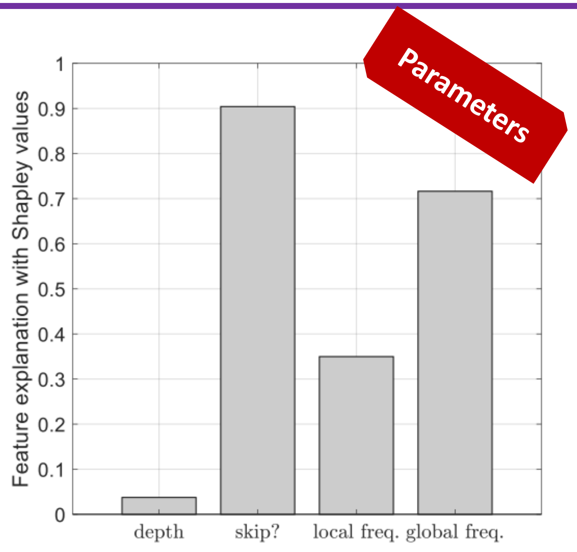
	1/8		
	Precision	Recall	F1
Handcrafted			
DCT NN	35.82	40.41	37.98
DCT SVM	95.00	94.98	94.99
DWT NN	49.98	99.72	66.59
DWT SVM	96.75	93.61	95.15
ScatterNet NN	86.12	77.46	81.56
ScatterNet SVM	86.75	91.67	89.14
Learning			
SimpleNet	75.76	36.49	49.25
ResNet	82.11	86.66	84.32
DenseNet	89.67	90.34	90.01
InceptionNet	83.39	79.15	81.22
Forensic			
CNN Spot	66.57	82.93	73.74
F3Net	80.43	79.54	79.67
Ours			
BGI NN	94.67	96.40	95.39
BGI SVM	94.59	97.73	96.09

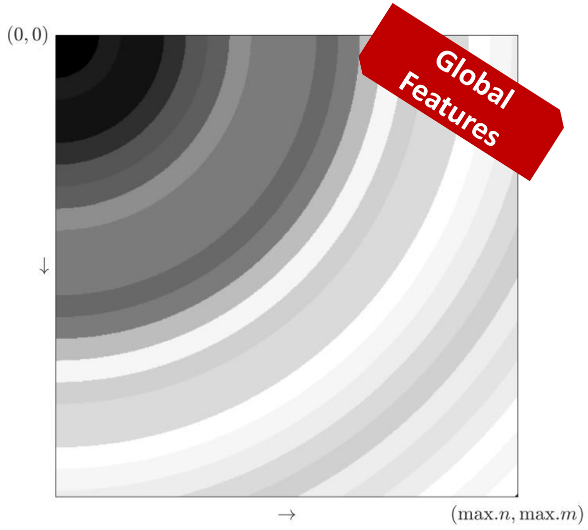
AIGC Detection: Geometric and Signal Robustness

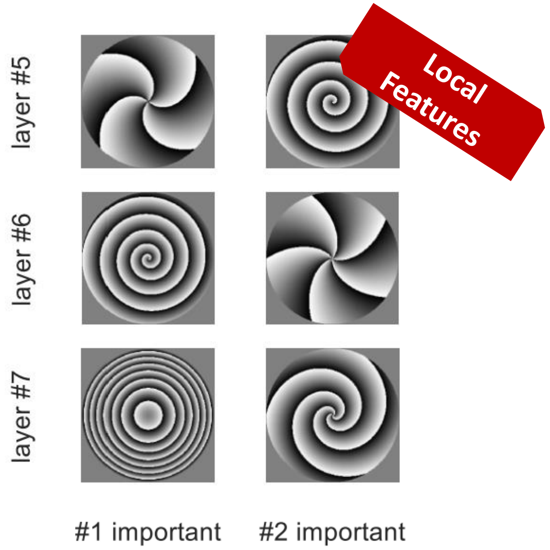


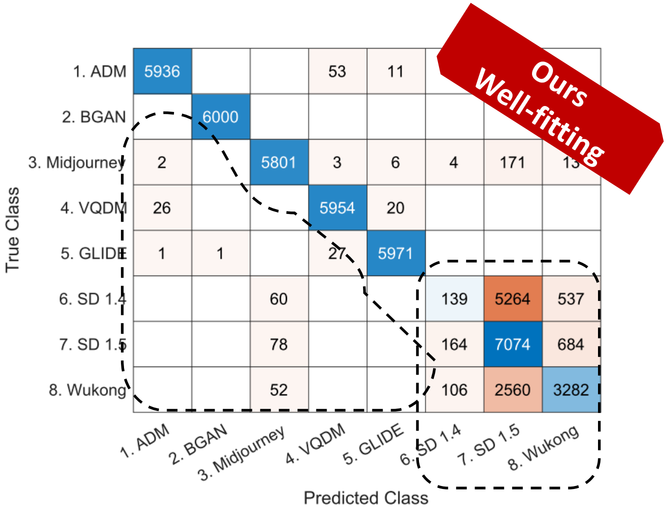
	Geometric Degradation Precision	Geometric Degradation Recall	Geometric Degradation F1	Signal Degradation Precision	Signal Degradation Recall	Signal Degradation F1
Handcrafted						
DCT NN	0	0	0	0	0	0
DCT SVM	80.86	95.03	87.38	78.50	91.35	84.44
DWT NN	50.62	25.59	33.99	52.40	26.11	34.86
DWT SVM	79.70	94.75	86.58	81.47	96.65	88.41
ScatterNet NN	69.34	88.58	77.79	67.97	95.97	79.58
ScatterNet SVM	90.67	80.23	85.13	92.37	90.38	91.36
Learning						
SimpleNet	65.03	85.90	74.02	66.13	92.61	77.16
ResNet	91.70	83.85	87.60	94.54	89.56	91.98
DenseNet	96.02	89.92	92.87	98.78	90.01	94.19
InceptionNet	92.00	92.24	92.12	96.77	84.06	89.97
Forensic						
CNN Spot	83.12	80.64	81.51	68.35	59.32	63.14
F3Net	79.83	77.57	77.96	80.96	74.83	77.13
Ours						
BGI NN	96.84	92.01	94.36	90.03	95.10	92.50
BGI SVM	96.45	93.40	94.90	92.52	95.84	94.15

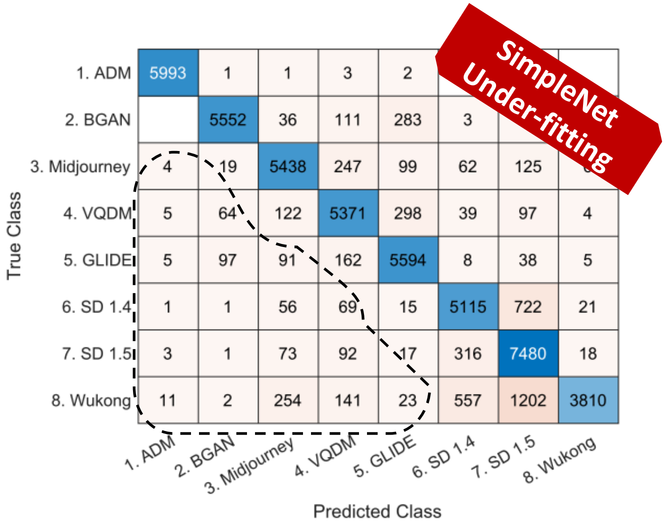
AIGC Detection: Visualization and Interpretability

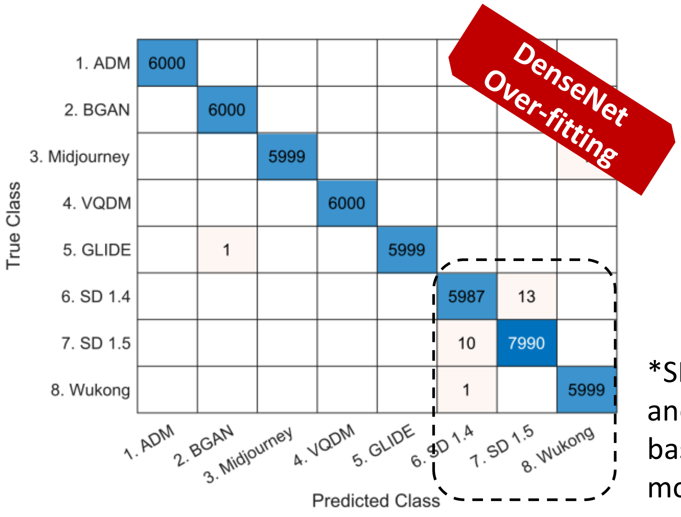






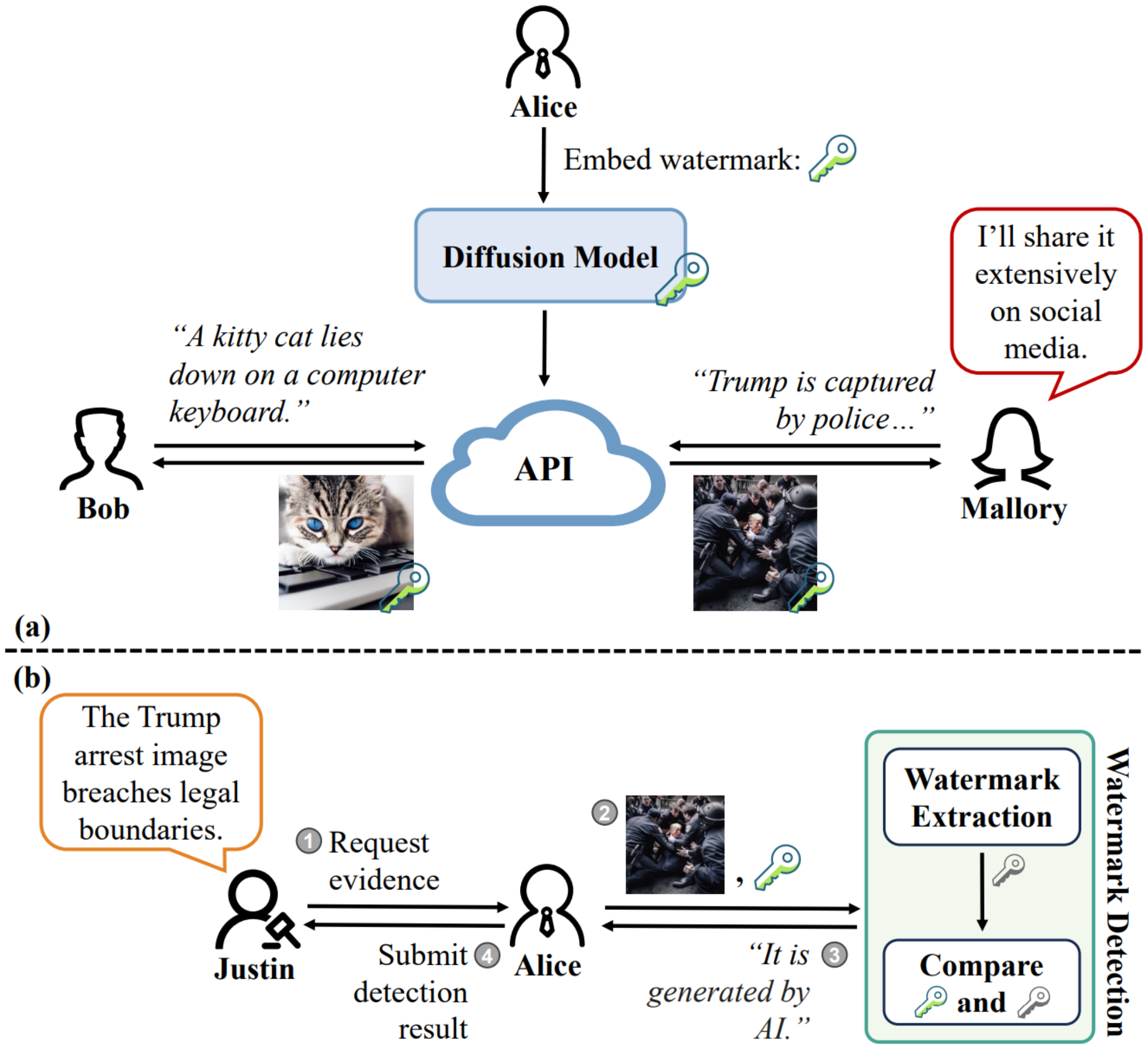


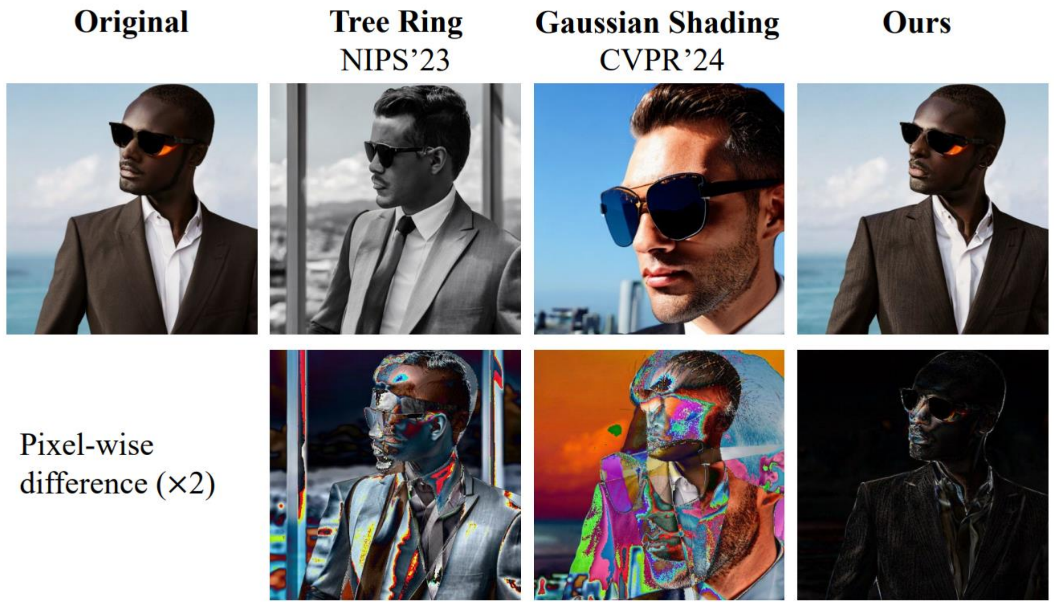




*SD 1.4, SD 1.5, and Wukong are based on the same model structure

AIGC Watermarking: Motivations

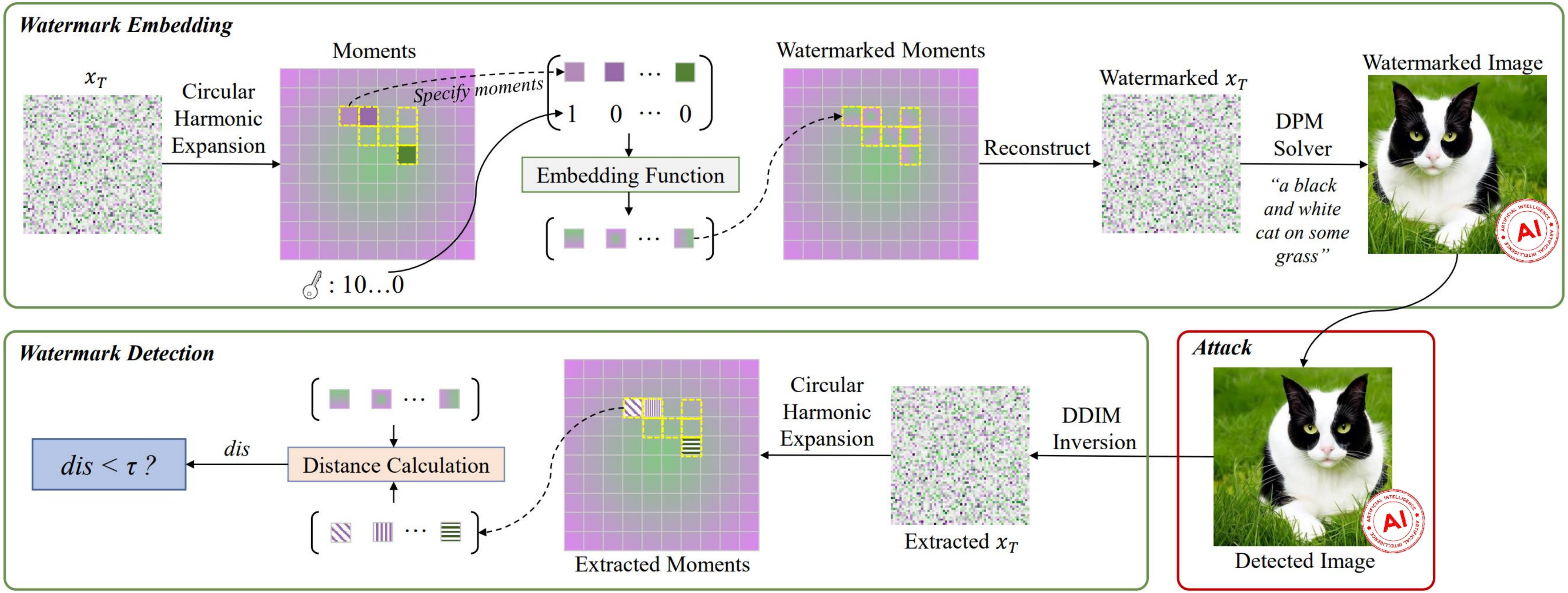




Is there a balance between robustness and imperceptibility?

- Y. Zhang, M. Shen, S. Qi*, et al. Embedding Robust Yet Imperceptible Watermarks in Diffusion Models: A Plug-and-Play AIGC Detector. *TIFS under review*, 2025.

AIGC Watermarking: Ideas



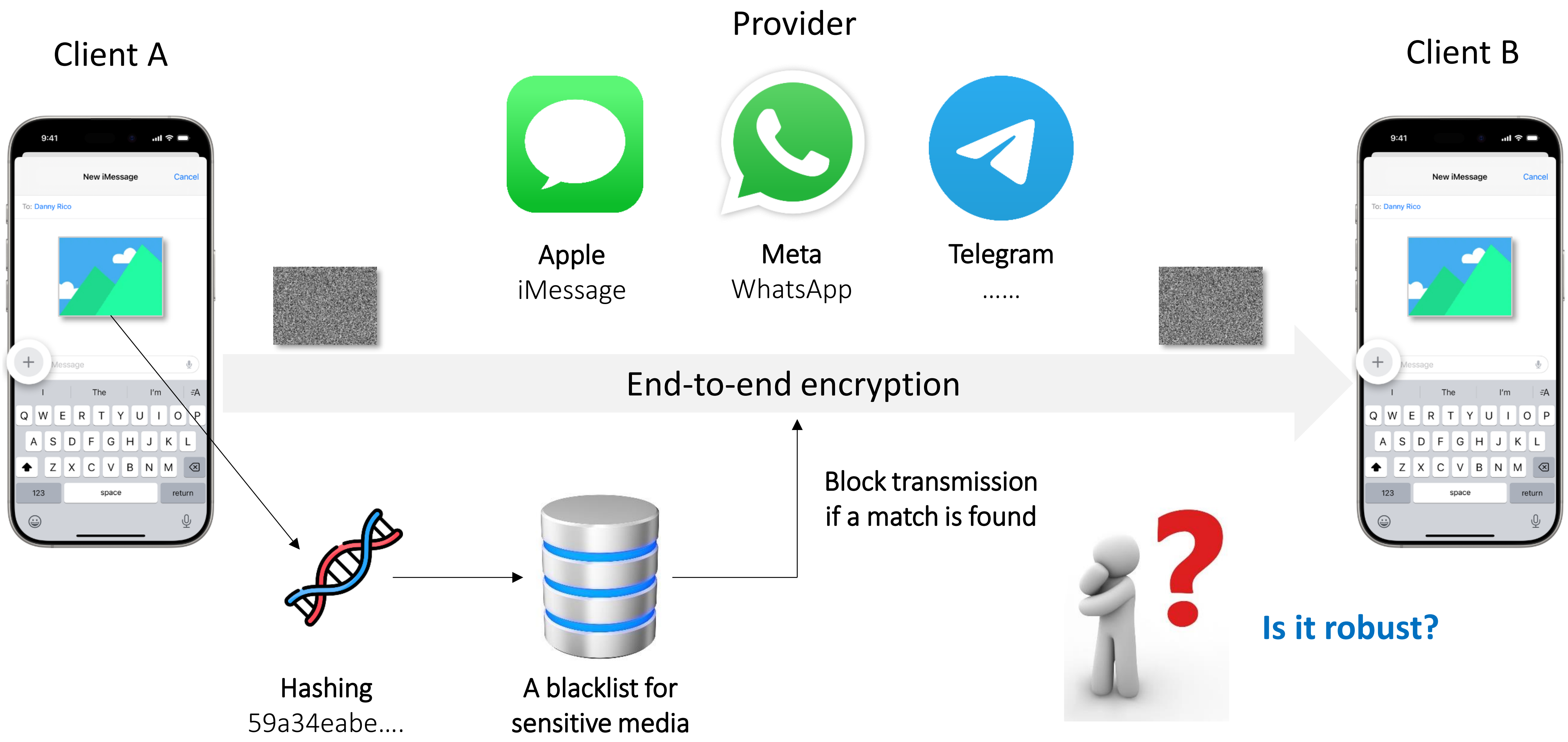
AIGC Watermarking: Robustness and Imperceptibility

Method	VAE based Bmshj'18	VAE based Cheng'20	DM based SDv2.1	Average
Pixel-level				*0.165*
DwtDct	0.005	0.002	0.003	0.003
DwtDctSvd	0.103	0.124	0.230	0.152
RivaGAN	0.014	0.017	0.123	0.051
Stable Signature	0.541	0.813	0.003	0.452
Content-level				*0.987*
Tree Ring	0.976	0.993	0.943	0.971
Gaussian Shading	1.000	1.000	1.000	1.000
Ours	0.990	0.983	1.000	0.991

Method	SSIM↑	LPIPS↓	WO-FID↓
Tree Ring	0.47	0.50	43.81
Gaussian Shading	0.20	0.74	48.32
Ours ($\alpha_2 = 0.02$)	**0.75**	**0.20**	**26.50**
Ours ($\alpha_2 = 0.04$)	0.62*	0.31*	35.02*

Tree Ring	Gaussian Shading	Ours ($\alpha_2 = 0.02$)

AIGC Hashing: Motivations



- Y. Zhang, Y. Sun, S. Qi*, et al. Atkscopes: Multiresolution Adversarial Perturbation as a Unified Attack on Perceptual Hashing and Beyond. *USENIX Security*, 2025.

AIGC Hashing: Ideas

Definition 1. ***(Multiresolution perturbation).*** *The addition of multiresolution perturbation is defined as follows:*

$$X'_{(x,y)\in D_{uvw}} = \mathcal{F}^{-1}\left(\mathcal{F}(X)+\delta\right), \tag{3}$$

with notations of

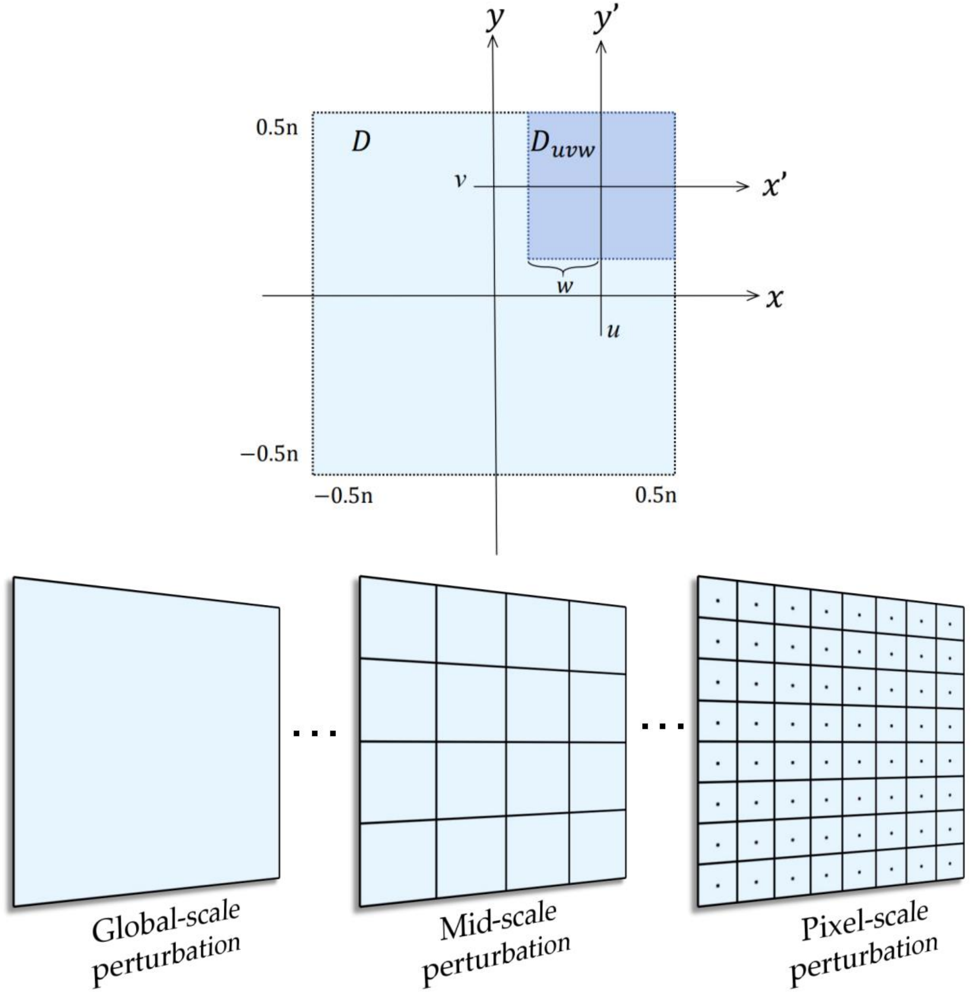
$$\mathcal{F}(X) = \langle X, V_{nm}^{uvw}\rangle = \iint_D (V_{nm}^{uvw}(x,y))^* X(x,y)dxdy, \tag{4}$$

and

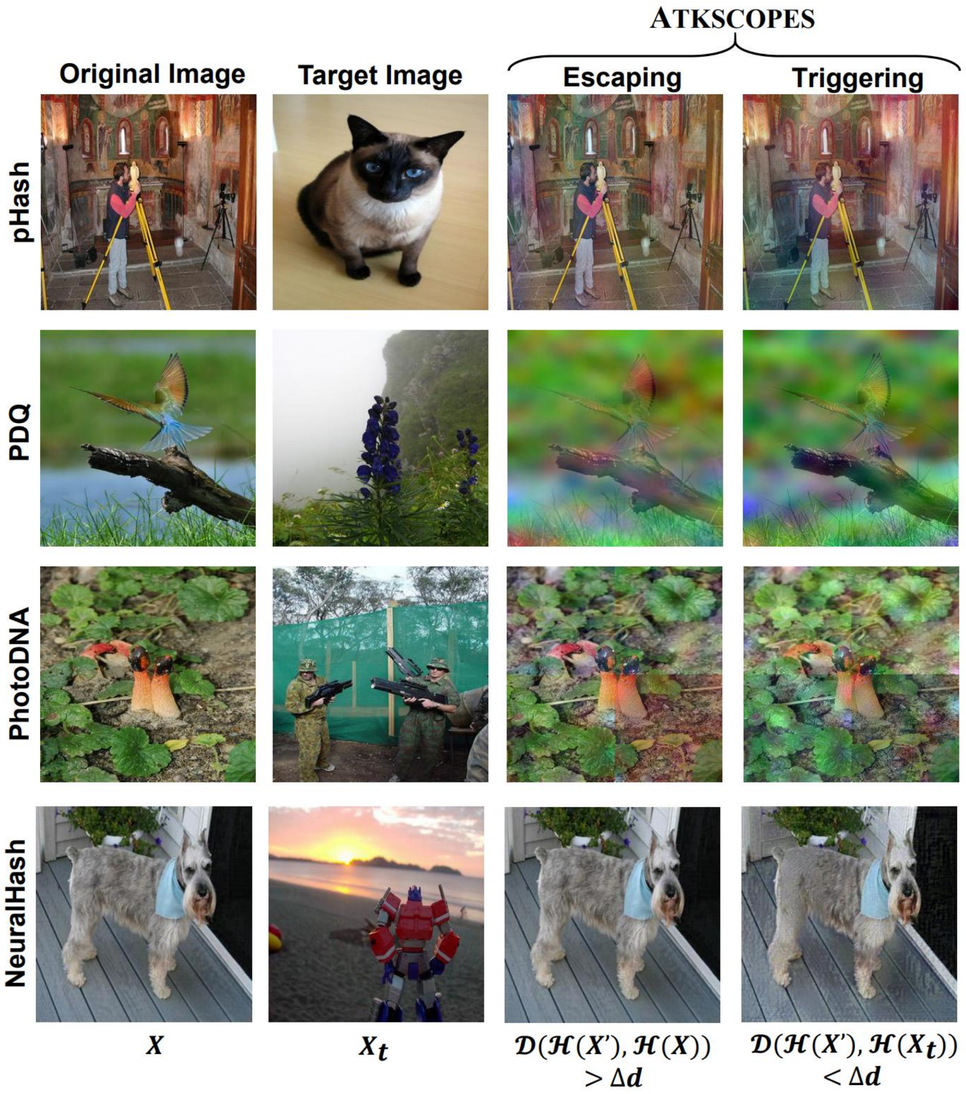
$$\mathcal{F}^{-1}(\mathcal{F}(X)) = \sum_{n,m} V_{nm}^{uvw}(x,y)\mathcal{F}(X), \tag{5}$$

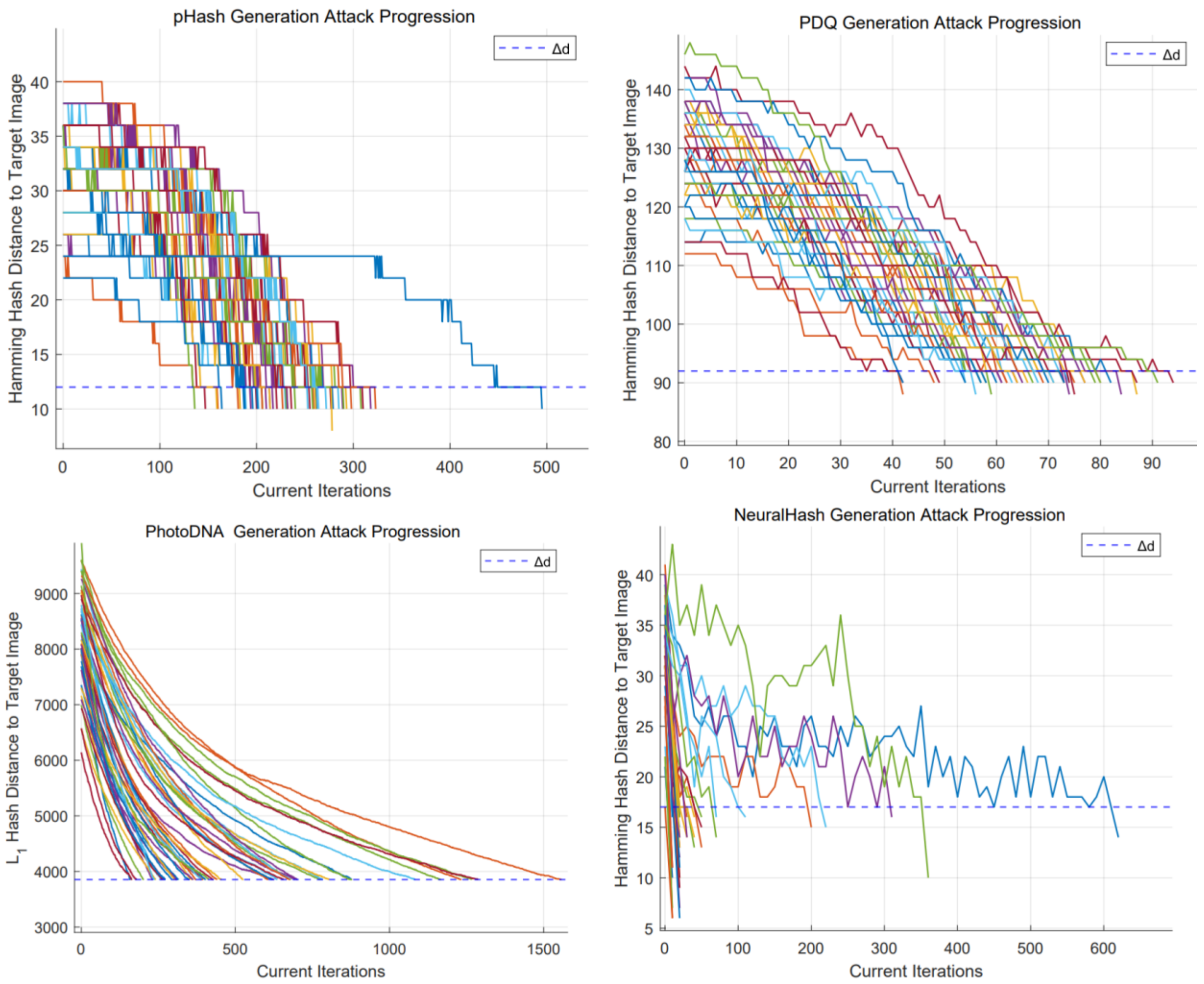
where $\mathcal{F}$ denotes the local orthogonal transformation [39], with image $X(x,y)$ on domain $(x,y)\in D$. The local orthogonal basis function V_{nm}^{uvw} is defined on the domain D_{uvw} with the order parameters $(n,m)\in\mathbb{Z}^2$, converting D to D_{uvw} by the translation offset (u,v) and the scaling factor w, as illustrated in Figure 2. Note that the local orthogonal basis function V_{nm}^{uvw} can be defined from any global orthogonal basis function V_{nm}, e.g., a family of harmonic functions, with following form:

$$V_{nm}^{uvw}(x,y) = V_{nm}(x',y') = V_{nm}\left(\frac{x-u}{w},\frac{y-v}{w}\right). \tag{6}$$



AIGC Hashing: Uniform, Fast, and Successful Attacks





Tutorial Outline

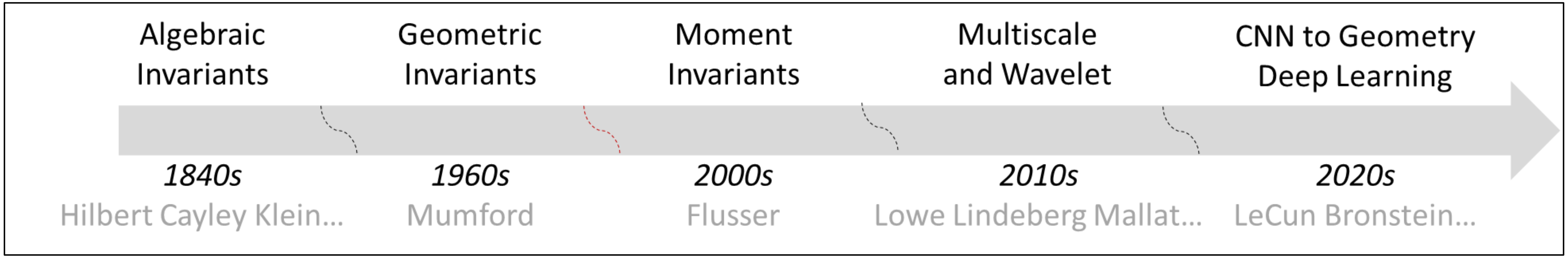
- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation

Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

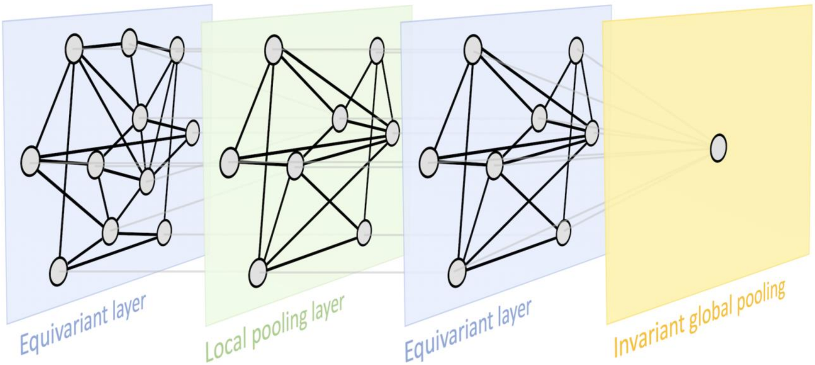
Conclusion 1: A Historical Perspective of Invariance

- A long history, from group theory, geometry, and physics
- In the era before deep learning: cornerstone
  - globally for the whole image (moment invariants), or locally for local parts of image (SIFT, DAISY, …).
- In the early era of deep learning: largely ignored
  - CNN vs. perceptron.
- In the era of rethinking deep learning: returned, geometric deep learning
  - locally and hierarchically (CNN, equivariant CNN, equivariant NN for group, set, graph…).

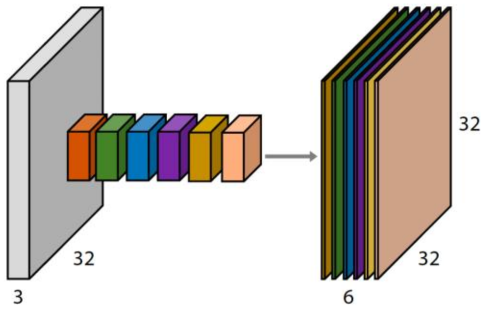


Conclusion 2: Rethinking Deep Learning by Invariance

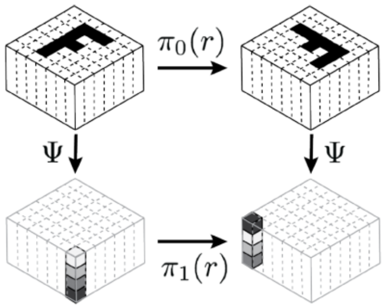
- Robust, interpretable and efficient (representation) learning
  - Perfect robustness, interpretable concept, and structural efficiency.
- CNN vs. perceptron on image data
  - Translation equi/in-variance.
- Geometric Deep Learning
  - For different transformations: wavelet scattering networks, group equivariant networks.
  - For different architectures and data types: deep sets/pointnet, graph networks, transformers.



Geometric deep learning blueprint

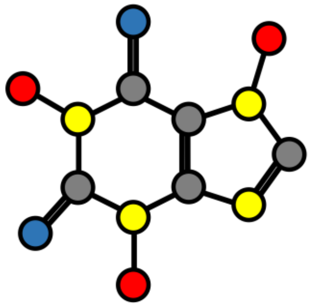


CNNs
translation



Group-CNNs
translation+rotation


DeepSets /Transformers
permutation


GNNs
permutation

Conclusion 3: Our Works for Invariance

Trustworthy AI as background

Symmetry priors in the natural world as principles

Expanding invariant representations at theoretical and practical levels

Invariant Representation

Invariant Theory
CSUR'23

- Moment Invariants → **Refining global invariants** TPAMI'24
- Multiscale and Wavelet → **Designing local invariants** TPAMI'23
- Geometric Deep Learning → **Exploring hierarchical invariants** ICCV'25

Forensic Tasks

AIGC Detection
TIP'25

AIGC Watermarking
TIP'25, TIFS review

AIGC Hashing
USENIX Security'25

Open Problem 1:
Exploring the Limits of Handcrafted Invariants

- **The Good:**
  - Embedding knowledge; good interpretability, robustness, and efficiency.
- **The Bad and The Ugly:**
  - Discriminability, adaptivity.
- **Open Problem:**
  - Upper bound of discriminability?
  - Data-driven learning, a must?
  - If for a specific task, handcrafted invariants always sufficient?
- **Research Opportunity:**
  - Overcomplete designs of invariants, e.g., time-frequency, multi-scale, hierarchical.
  - Feature selection and explanation, from over-complete to task-discriminative.

Open Problem 2:
More Flexible Designs for Learning Invariants

- **The Good:**
  - Discriminability, adaptivity.
- **The Bad and The Ugly:**
  - Limited invariance, inefficient implementation, especially for joint invariance.
- **Open Problem:**
  - Group convolution (symmetry sampling), uniformly good?
  - Element-wise operations and global pooling, sufficient for graphs/sets?
- **Research Opportunity:**
  - Continuous and high-order designs for local-equivariant and global-invariant representations.
  - Specific designs of equi/in-variance for different data types.

Open Problem 3: Real-world Impact and Application Considerations

- **The Good:**
  - Many low-level processing, some high-level tasks; AI for Science, e.g. AlphaFold.
- **The Bad and The Ugly:**
  - Real-world impact in broader applications.
- **Open Problem:**
  - Invariance, somewhat limit adaptivity?
  - Invariance, designed for generic tasks?
- **Research Opportunity:**
  - Designing high-performance invariants for specific tasks, i.e., specific data assumptions and knowledges.
  - Easy-to-use software, environment, and document.

There Is No Royal Road To Geometry

Tutorial Outline

- **Part 1:** Background and challenges (20 min)
- **Part 2:** Preliminaries of invariance (20 min)
- *Q&A / Break (10 min)*
- **Part 3:** Invariance in the era before deep learning (30 min)
- **Part 4:** Invariance in the early era of deep learning (10 min)
- *Q&A / Coffee Break (30 min)*
- **Part 5:** Invariance in the era of rethinking deep learning (50 min)
- **Part 6:** Conclusions and discussions (20 min)
- *Q&A (10 min)*

A Historical Perspective of Data Representation
Rethinking Deep Learning with Invariance: The Good, The Bad, and The Ugly

§ Thank you!

by Shuren Qi
shurenqi@cuhk.edu.hk | shurenqi.github.io